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Feynman propagator for a particle with arbitrary spin

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Abstract. Based on the solution to the Rarita–Schwinger equations, a direct derivation of the projection operator and propagator for a particle with arbitrary spin is worked out. The projection operator constructed by Behrends and Fronsdal is re-deduced and confirmed, and simplified in the case of half-integral spin; the general commutation rules and Feynman propagator for a free particle of any spin are derived, and explicit expressions for the propagators for spins 3/2, 2, 5/2, 3, 7/2, 4 are provided.

1 Introduction

In order to perform analyses for high energy processes such as

$$\begin{gathered} b_1(1235) \to \omega + \pi, \quad \bar{p}p(^3P_2) \to f_2(1270) + \pi, \\ a_3(2050) \to f_2(1270) + \pi, \quad H \to W^+W^-, \\ J/\Psi \to a_2(1320) + \rho, \end{gathered}$$

and so on, it is necessary to employ higher spin relativistic wave functions, projection operators and Feynman propagators [1–4]. Recently, a systematical method [5] was developed to solve the Rarita–Schwinger [6] equations and derive explicit positive and negative energy wave functions for higher spin particles. Based on this work, we have carried out a further investigation on the projection operator, the commutation rules and the Feynman propagator for a free particle with arbitrary spin. The results are reported in the present paper.

The concept of higher spin projection operators was first introduced by Behrends and Fronsdal [7,8] in 1957 when they undertook a calculation of the lifetimes and spectra of Fermi decays for higher spin particles. Based on the properties of these operators derived from the Klein–Gordon and Rarita–Schwinger equations, they constructed an explicit form of the projection operators for particles with arbitrary integral or half-integral spins. This construction was carried out first in the rest system and then generalized to an arbitrary frame. In 1965, Zemach [9] proposed an alternative way to construct this kind of projection operators in the rest frame. However, it has been found recently by Chung [1,2] and by Filippini et al. [4] that the Zemach formalism is incorrect because it is essentially a non-relativistic one. Considering that the B-F formulas are basically constructed in the rest frame, an independent check of their correctness might be necessary. A direct calculation of these projection operators based on the explicit expressions of the wave functions and performed in an arbitrary frame should yield a reliable check. The first part of the present work will be devoted to this check. The results show that the B-F formulas are correct. It is found that the projection operators for half-integral spins should be derived in a way different compared with that for integral spins, because Dirac γ matrices are involved in this case. They are derived by virtue of a set of newly found sum relations about γ matrices and spins 1/2 and 1 wave functions, and are simplified such that it is suitable for calculation of the Feynman propagators.

The propagator for a free particle of arbitrary spin j was first studied by Weinberg [10] in 1964; the treatment is based on a $2j+1$-component field $\varphi_m(x)(m = j, j-1, \cdots, -j)$ constructed from the $2j+1$-dimensional unitary representation of the boost operation. The propagator is defined as the vacuum expectation value of time-ordered field component operators, $S^{(j)}_{mm'}(x-y) = \langle T\{\varphi_m(x)\varphi^+_{m'}(y)\}\rangle_0$. It is emphasized by Weinberg in this work that for particles with spin $j \geq 1$, there appear extra non-covariant terms in the propagator (called the "raw" propagator), and that the cure to this problem would be to add non-covariant "contact" terms to the Hamiltonian in such a way as to cancel out these non-covariant terms, so that the "true" propagator used in the Feynman rules contains only the covariant part. In 1968, Scadron [11] calculated the high-spin propagator in a different formalism. The wave functions used in this calculation are tensors or tensor-spinors that are constructed by Auvil and Brehm [9] and are re-derived in our previous work [5]. In the Scadron approach, however, it is the contracted propagator (numerator) that was studied, which is defined as a contraction of the spin sum, $\sum_m e_m^{\mu_1\mu_2\cdots\mu_n}(K)\bar{e}_m^{\nu_1\nu_2\cdots\nu_n}(K)$, with the initial momenta $p^{\nu_1}p^{\nu_2}\cdots p^{\nu_n}$ and the final momenta $p'^{\mu_1}p'^{\mu_2}\cdots p'^{\mu_n}$. In 1992, based on the work of Weinberg [10], Ahluwalia and Ernst [12] suggested that the high-spin propagators can be constructed as that of spin 1/2. Their definition of the propagator (different from that of Weinberg) is $S^{(j)}(x-y) = \langle T\{\Psi^{(j)}(x)\bar{\Psi}^{(j)}(y)\}\rangle_0$, the vacuum expectation value of time-ordered field operators $\Psi^{(j)}(x)$, which are the spin sums of the field components, $\Psi^{(j)}(x) = \sum_m \psi_m(x)$. However, no explicit expressions for the propagators were derived since the spin sum of the spinors is not worked out. In the second part of this paper, we shall choose the field operators $\Psi^{(j)}(x)$ as the solutions to the Rarita–Schwinger equations [5,6] to calculate the propagators defined as the vacuum expectation value of time-ordered field operators. We shall work out the covariant part of the Feynman propagator as well as the extra non-covariant terms for a particle with arbitrary spin. Although only the covariant part will be used in the Feynman rules, the non-covariant terms might serve as guides to the construction of the Hamiltonian as pointed out by Weinberg [10]. It is found that there is a new kind of extra non-covariant terms originating from the γ factors in the expression of the propagator for a half-integral spin, besides that for an integral spin. These terms will be derived in a step-by-step way. Especially, explicit expressions for the propagators for spins 3/2, 2, 5/2, 3, 7/2 and 4, which are often useful for experimentalists, are worked out in detail. Our formulas for the propagators are different from those derived by Weinberg [10] because the definitions are different and the field operators are also different, while by contracting our propagators with the initial and final momenta, a contracted propagator consistent with that derived by Scadron [11] could be produced because we used the same spin sums of tensors or tensor-spinors that satisfy the Rarita–Schwinger equations. Such an equivalence, however, will not be discussed in the present work.

2 The projection operator

2.1 Integral spin

For a particle with arbitrary integral spin n and rest-mass W, the wave functions could be expressed as

$$A^{\nu_1\nu_2\cdots\nu_n}(x) = \sum_{\boldsymbol{p}}\sum_{m=-n}^{n}\frac{1}{\sqrt{2EV}}\Big[a_m(\boldsymbol{p})e_m^{\nu_1\nu_2\ldots\nu_n}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px} + b_m^+(\boldsymbol{p})\bar{e}_m^{\nu_1\nu_2\ldots\nu_n}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}\Big], \tag{1a}$$

where $E=\sqrt{\boldsymbol{p}^2+W^2}$, $p=(\boldsymbol{p},\mathrm{i}E)$, $e^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$ and $\bar{e}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$ are, respectively, the positive and negative energy wave functions in the momentum representations and satisfy the wave equation

$$(p^2+W^2)A^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p}) = 0, \tag{1b}$$

and the subsidiary conditions

$$A^{\nu_1\nu_2\cdots\nu_i\cdots\nu_j\cdots\nu_n}(\boldsymbol{p}) = A^{\nu_1\nu_2\cdots\nu_j\cdots\nu_i\cdots\nu_n}(\boldsymbol{p}), \tag{1c}$$
$$p_\nu A^{\nu\nu_2\cdots\nu_i\cdots\nu_j\cdots\nu_n}(\boldsymbol{p}) = 0, \tag{1d}$$
$$A^{\nu\nu\nu_3\cdots\nu_n}(\boldsymbol{p}) = 0; \tag{1e}$$

here $A^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$ stands for $e^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$ or $\bar{e}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$. The solution to (1b)–(1e) has been carried out previously and the results can be written as [5]

$$e_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p}) = \sum_{\lambda_n=-1}^{1}\langle n-1, m-\lambda_n, 1, \lambda_n | n-1, 1, n, m\rangle \times e_{m-\lambda_n}^{\nu_1\nu_2\cdots\nu_{n-1}}(\boldsymbol{p})e_{\lambda_n}^{\nu_n}(\boldsymbol{p}), \tag{2a}$$

$$\bar{e}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p}) = \sum_{\lambda_n=-1}^{1}\langle n-1, m-\lambda_n, 1, \lambda_n | n-1, 1, n, m\rangle \times \bar{e}_{m-\lambda_n}^{\nu_1\nu_2\cdots\nu_{n-1}}(\boldsymbol{p})\bar{e}_{\lambda_n}^{\nu_n}(\boldsymbol{p}), \tag{2b}$$

where $e_\lambda^\nu(\boldsymbol{p})$ and $\bar{e}_\lambda^\nu(\boldsymbol{p})$ are the positive and negative energy wave functions for spin 1 and are related by

$$\bar{e}_{\lambda_i}^{\nu_i}(\boldsymbol{p}) = g_{\nu_i\mu_i}\left(e_{\lambda_i}^{\mu_i}(\boldsymbol{p})\right)^* = (-1)^{\lambda_i}e_{-\lambda_i}^{\nu_i}(\boldsymbol{p}),$$
$$g_{\nu_i\mu_i} = \mathrm{diag}\{1,1,1,-1\}. \tag{2c}$$

The wave functions $e^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$ and $\bar{e}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$ are normalized according to

$$e_m^{\nu_1\nu_2\ldots\nu_n}(\boldsymbol{p})\bar{e}_{m'}^{\nu_1\nu_2\ldots\nu_n}(\boldsymbol{p}) = \delta_{m,m'}. \tag{2d}$$

As in the case of spin 1, the projection operator for spin n is defined as

$$P^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(n,p) = \sum_{m=-n}^{n} e_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\bar{e}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p}). \tag{3}$$

From (1b)–(1e) and the normalization condition (2d), it is easy to find that the projection operator possesses the following properties [7,8]:

$$P^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_i\cdots\nu_j\cdots\nu_n}(n,p) = P^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_j\cdots\nu_i\cdots\nu_n}(n,p), \tag{4a}$$
$$p_\nu P^{\mu_1\mu_2\cdots\mu_n\nu\nu_2\cdots\nu_n}(n,p) = 0, \tag{4b}$$
$$P^{\mu_1\mu_2\cdots\mu_n\nu\nu\nu_3\cdots\nu_n}(n,p) = 0, \tag{4c}$$
$$P^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(n,p)P^{\nu_1\nu_2\cdots\nu_n\varepsilon_1\varepsilon_2\cdots\varepsilon_n}(n,p) = P^{\mu_1\mu_2\cdots\mu_n\varepsilon_1\varepsilon_2\cdots\varepsilon_n}(n,p). \tag{4d}$$

By using the explicit expressions for $e^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$ and $\bar{e}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$, and the projection operator for spin 1 given by

$$P^{\mu_1\nu_1}(p) = \sum_{\lambda=-1}^{1} e_\lambda^{\mu_1}(\boldsymbol{p})\bar{e}_\lambda^{\nu_1}(\boldsymbol{p}) = \sum_{\lambda=-1}^{1}\bar{e}_\lambda^{\mu_1}(\boldsymbol{p})e_\lambda^{\nu_1}(\boldsymbol{p}) = \delta_{\mu_1\nu_1} + \frac{p_{\mu_1}p_{\nu_1}}{W^2}, \tag{5}$$

one can perform a direct calculation of the projection operator for a given integral spin. For example, in the case of spin 2, by utilizing the explicit form of the positive energy wave functions listed below:

$$e_2^{\nu_1\nu_2} = e_{+1}^{\nu_1}e_{+1}^{\nu_2}, \quad e_1^{\nu_1\nu_2} = \frac{1}{\sqrt{2}}[e_{+1}^{\nu_1}e_0^{\nu_2} + e_0^{\nu_1}e_{+1}^{\nu_2}],$$
$$e_0^{\nu_1\nu_2} = \frac{1}{\sqrt{6}}[e_{+1}^{\nu_1}e_{-1}^{\nu_2} + 2e_0^{\nu_1}e_0^{\nu_2} + e_{-1}^{\nu_1}e_{+1}^{\nu_2}],$$
$$e_{-1}^{\nu_1\nu_2} = \frac{1}{\sqrt{2}}[e_0^{\nu_1}e_{-1}^{\nu_2} + e_{-1}^{\nu_1}e_0^{\nu_2}], \quad e_{-2}^{\nu_1\nu_2} = e_{-1}^{\nu_1}e_{-1}^{\nu_2},$$

and the corresponding negative energy wave functions $\bar{e}_m^{\mu_1\mu_2}$ (which can be listed by replacing e_λ^ν with $\bar{e}_\lambda^\mu$ in the above expressions), and noticing that $\bar{e}_\lambda^\nu = (-1)^\lambda e_{-\lambda}^\nu$, a straightforward calculation leads to

$$\begin{aligned} P^{\mu_1\mu_2\nu_1\nu_2}(2,p) &= \sum_{m=-2}^{2} \bar{e}_m^{\mu_1\mu_2} e_m^{\nu_1\nu_2} \\ &= \frac{1}{2}\sum_\lambda \bar{e}_\lambda^{\mu_1}e_\lambda^{\nu_1}\sum_{\lambda'}\bar{e}_{\lambda'}^{\mu_2}e_{\lambda'}^{\nu_2} + \frac{1}{2}\sum_\lambda \bar{e}_\lambda^{\mu_1}e_\lambda^{\nu_2}\sum_{\lambda'}\bar{e}_{\lambda'}^{\mu_2}e_{\lambda'}^{\nu_1} \\ &\quad -\frac{1}{3}\sum_\lambda \bar{e}_\lambda^{\mu_1}e_\lambda^{\mu_2}\sum_{\lambda'}\bar{e}_{\lambda'}^{\nu_1}e_{\lambda'}^{\nu_2} \\ &= \frac{1}{2}P^{\mu_1\nu_1}P^{\mu_2\nu_2} + \frac{1}{2}P^{\mu_1\nu_2}P^{\mu_2\nu_1} - \frac{1}{3}P^{\mu_1\mu_2}P^{\nu_1\nu_2}, \end{aligned} \quad (6a)$$

or alternatively, by using $P^{\mu_i\nu_i} = P^{\nu_i\mu_i}$,

$$P^{\mu_1\mu_2\nu_1\nu_2}(2,p) \quad (6b)$$
$$= \frac{1}{4}\sum_{\substack{P(\mu_1\mu_2)\\P(\nu_1\nu_2)}}\left[P^{\mu_1\nu_1}(p)P^{\mu_2\nu_2}(p) - \frac{1}{3}P^{\mu_1\mu_2}(p)P^{\nu_1\nu_2}(p)\right],$$

where the sum is over all permutations of the μ's and ν's. Similarly, in the case of spin 3, by utilizing the following explicit form of the positive energy wave functions:

$$\begin{aligned} e_3^{\nu_1\nu_2\nu_3} &= e_{+1}^{\nu_1}e_{+1}^{\nu_2}e_{+1}^{\nu_3}, \\ e_2^{\nu_1\nu_2\nu_3} &= \frac{1}{\sqrt{3}}[e_{+1}^{\nu_1}e_{+1}^{\nu_2}e_0^{\nu_3} + e_{+1}^{\nu_1}e_0^{\nu_2}e_{+1}^{\nu_3} + e_0^{\nu_1}e_{+1}^{\nu_2}e_{+1}^{\nu_3}], \\ e_1^{\nu_1\nu_2\nu_3} &= \frac{1}{\sqrt{15}}[e_{+1}^{\nu_1}e_{+1}^{\nu_2}e_{-1}^{\nu_3} + e_{+1}^{\nu_1}e_{-1}^{\nu_2}e_{+1}^{\nu_3} + e_{-1}^{\nu_1}e_{+1}^{\nu_2}e_{+1}^{\nu_3} \\ &\quad +2e_{+1}^{\nu_1}e_0^{\nu_2}e_0^{\nu_3} + 2e_0^{\nu_1}e_{+1}^{\nu_2}e_0^{\nu_3} + 2e_0^{\nu_1}e_0^{\nu_2}e_{+1}^{\nu_3}], \\ e_0^{\nu_1\nu_2\nu_3} &= \frac{1}{\sqrt{10}}[e_{+1}^{\nu_1}e_0^{\nu_2}e_{-1}^{\nu_3} + e_0^{\nu_1}e_{+1}^{\nu_2}e_{-1}^{\nu_3} + e_{+1}^{\nu_1}e_{-1}^{\nu_2}e_0^{\nu_3} \\ &\quad +2e_0^{\nu_1}e_0^{\nu_2}e_0^{\nu_3} + e_{-1}^{\nu_1}e_{+1}^{\nu_2}e_0^{\nu_3} \\ &\quad +e_0^{\nu_1}e_{-1}^{\nu_2}e_{+1}^{\nu_3} + e_{-1}^{\nu_1}e_0^{\nu_2}e_{+1}^{\nu_3}], \\ e_{-1}^{\nu_1\nu_2\nu_3} &= \frac{1}{\sqrt{15}}[2e_0^{\nu_1}e_0^{\nu_2}e_{-1}^{\nu_3} \\ &\quad +2e_0^{\nu_1}e_{-1}^{\nu_2}e_0^{\nu_3} + 2e_{-1}^{\nu_1}e_0^{\nu_2}e_0^{\nu_3} + e_{+1}^{\nu_1}e_{-1}^{\nu_2}e_{-1}^{\nu_3} \\ &\quad +e_{-1}^{\nu_1}e_{+1}^{\nu_2}e_{-1}^{\nu_3} + e_{-1}^{\nu_1}e_{-1}^{\nu_2}e_{+1}^{\nu_3}], \\ e_{-2}^{\nu_1\nu_2\nu_3} &= \frac{1}{\sqrt{3}}[e_0^{\nu_1}e_{-1}^{\nu_2}e_{-1}^{\nu_3} + e_{-1}^{\nu_1}e_0^{\nu_2}e_{-1}^{\nu_3} + e_{-1}^{\nu_1}e_{-1}^{\nu_2}e_0^{\nu_3}], \\ e_{-3}^{\nu_1\nu_2\nu_3} &= e_{-1}^{\nu_1}e_{-1}^{\nu_2}e_{-1}^{\nu_3} \end{aligned}$$

and the corresponding negative energy wave functions, one finds by direct calculation

$$\begin{aligned} P^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(3,p) &= \sum_{m=-3}^{3}\bar{e}_m^{\mu_1\mu_2\mu_3}e_m^{\nu_1\nu_2\nu_3} \\ &= \frac{1}{6}[P^{\mu_1\nu_1}P^{\mu_2\nu_2}P^{\mu_3\nu_3} + P^{\mu_1\nu_1}P^{\mu_2\nu_3}P^{\mu_3\nu_2} \\ &\quad +P^{\mu_1\nu_2}P^{\mu_2\nu_1}P^{\mu_3\nu_3} + P^{\mu_1\nu_2}P^{\mu_2\nu_3}P^{\mu_3\nu_1} \\ &\quad +P^{\mu_1\nu_3}P^{\mu_2\nu_1}P^{\mu_3\nu_2} + P^{\mu_1\nu_3}P^{\mu_2\nu_2}P^{\mu_3\nu_1}] \\ &\quad -\frac{1}{15}[P^{\mu_1\mu_2}P^{\nu_1\nu_2}P^{\mu_3\nu_3} + P^{\mu_1\mu_3}P^{\nu_1\nu_2}P^{\mu_2\nu_3} \\ &\quad +P^{\mu_2\mu_3}P^{\nu_1\nu_2}P^{\mu_1\nu_3} + P^{\mu_1\mu_2}P^{\nu_1\nu_3}P^{\mu_3\nu_2} \\ &\quad +P^{\mu_1\mu_3}P^{\nu_1\nu_3}P^{\mu_2\nu_2} + P^{\mu_2\mu_3}P^{\nu_1\nu_3}P^{\mu_1\nu_2} \\ &\quad +P^{\mu_1\mu_2}P^{\nu_2\nu_3}P^{\mu_3\nu_1} + P^{\mu_1\mu_3}P^{\nu_2\nu_3}P^{\mu_2\nu_1} \\ &\quad +P^{\mu_2\mu_3}P^{\nu_2\nu_3}P^{\mu_1\nu_1}], \end{aligned} \quad (7a)$$

or alternatively,

$$P^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(3,p) \quad (7b)$$
$$= \frac{1}{36}\sum_{\substack{P(\mu_1\mu_2\mu_3)\\P(\nu_1\nu_2\nu_3)}}\left[P^{\mu_1\nu_1}P^{\mu_2\nu_2}P^{\mu_3\nu_3} - \frac{3}{5}P^{\mu_1\mu_2}P^{\nu_1\nu_2}P^{\mu_3\nu_3}\right].$$

This method could be generalized to the case of spin $4, 5, \cdots$ The expressions given by (6b) and (7b) and so on are exactly the same as the ones constructed by Behrends and Fronsdal [7,8], which can be written, in our notation,

$$P^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(n,p) \quad (8)$$
$$\begin{aligned} &= \left(\frac{1}{n!}\right)^2\sum_{\substack{P(\mu)\\P(\nu)}}\Bigg[\prod_{i=1}^{n}P^{\mu_i\nu_i} + A_1P^{\mu_1\mu_2}P^{\nu_1\nu_2}\prod_{i=3}^{n}P^{\mu_i\nu_i} + \cdots \\ &\quad +A_rP^{\mu_1\mu_2}P^{\nu_1\nu_2}P^{\mu_3\mu_4}P^{\nu_3\nu_4}\cdots \\ &\quad P^{\mu_{(2r-1)}\mu_{2r}}P^{\nu_{(2r-1)}\nu_{2r}}\prod_{i=2r+1}^{n}P^{\mu_i\nu_i} + \cdots \\ &\quad + \begin{cases} A_{n/2}P^{\mu_1\mu_2}P^{\nu_1\nu_2}\cdots P^{\mu_{n-1}\mu_n}P^{\nu_{n-1}\nu_n} \\ \qquad \text{(for even } n\text{)} \\ A_{(n-1)/2}P^{\mu_1\mu_2}P^{\nu_1\nu_2}\cdots P^{\mu_{n-2}\mu_{n-1}}P^{\nu_{n-2}\nu_{n-1}}P^{\mu_n\nu_n} \\ \qquad \text{(for odd } n\text{)} \end{cases}\Bigg], \end{aligned}$$

with

$$A_r(n) \quad (9)$$
$$= \left(-\frac{1}{2}\right)^r\frac{n!}{r!(n-2r)!(2n-1)(2n-3)\cdots(2n-2r+1)}.$$

Thus the B-F projection operator for integral spin is confirmed.

2.2 Half-integral spin

For a particle with arbitrary half-integral spin $n+1/2$ (n is an integral), the wave function may be expressed by [5]

$$\Psi^{\nu_1\nu_2\cdots\nu_n}(x)=\frac{1}{\sqrt{V}}\sum_{\boldsymbol{p}}\sum_{m=-(n+1/2)}^{n+1/2}\Big[a_m(\boldsymbol{p})U_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}+b_m^+(\boldsymbol{p})V_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}\Big], \tag{10}$$

where $U^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$ and $V^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$ are, respectively, the positive and negative energy wave functions in momentum representations and take the following form:

$$U_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})=\sqrt{\frac{n+\frac{1}{2}+m}{2n+1}}e_{m-\frac{1}{2}}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})u_{\frac{1}{2}}(\boldsymbol{p})+\sqrt{\frac{n+\frac{1}{2}-m}{2n+1}}e_{m+\frac{1}{2}}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})u_{-\frac{1}{2}}(\boldsymbol{p}), \tag{11a}$$

$$V_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})=\sqrt{\frac{n+\frac{1}{2}+m}{2n+1}}\bar{e}_{m-\frac{1}{2}}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})v_{\frac{1}{2}}(\boldsymbol{p})+\sqrt{\frac{n+\frac{1}{2}-m}{2n+1}}\bar{e}_{m+\frac{1}{2}}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})v_{-\frac{1}{2}}(\boldsymbol{p}), \tag{11b}$$

with $u_r(\boldsymbol{p})$ and $v_r(\boldsymbol{p})(r=\pm1/2)$ the usual Dirac 1/2 spinors. The adjoint field of $\Psi^{\nu_1\nu_2\cdots\nu_n}(x)$ is as in the case of spin 1/2 defined by

$$\bar{\Psi}^{\nu_1\nu_2\cdots\nu_n}(x)=(\Psi^{\nu_1\nu_2\cdots\nu_n}(x))^+\gamma_4 =\frac{1}{\sqrt{V}}\sum_{\boldsymbol{p}m}\Big[a_m^+(\boldsymbol{p})\bar{U}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}+b^{(}_m\boldsymbol{p})\bar{V}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}\Big], \tag{12}$$

where

$$\bar{U}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})=\sqrt{\frac{n+\frac{1}{2}+m}{2n+1}}\bar{e}_{m-\frac{1}{2}}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\bar{u}_{\frac{1}{2}}(\boldsymbol{p})+\sqrt{\frac{n+\frac{1}{2}-m}{2n+1}}\bar{e}_{m+\frac{1}{2}}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\bar{u}_{-\frac{1}{2}}(\boldsymbol{p}), \tag{13a}$$

$$\bar{V}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})=\sqrt{\frac{n+\frac{1}{2}+m}{2n+1}}e_{m-\frac{1}{2}}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\bar{v}_{\frac{1}{2}}(\boldsymbol{p})+\sqrt{\frac{n+\frac{1}{2}-m}{2n+1}}e_{m+\frac{1}{2}}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\bar{v}_{-\frac{1}{2}}(\boldsymbol{p}), \tag{13b}$$

$$\bar{u}_r(\boldsymbol{p})=u_r^+(\boldsymbol{p})\gamma_4,\quad \bar{v}_r(\boldsymbol{p})=v_r^+(\boldsymbol{p})\gamma_4. \tag{13c}$$

These wave functions are normalized according to

$$\bar{U}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})U_{m'}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p}) =-(W/E)\delta_{mm'}=\bar{V}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})V_{m'}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p}), \tag{14a}$$

$$\bar{U}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})V_{m'}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p}) =\bar{V}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})U_{m'}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})=0. \tag{14b}$$

In a similar fashion as in the case of spin 1/2, the projection operator for spin $n+1/2$ is defined by

$$P_+^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(p) =\sum_m U_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\bar{U}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p}), \tag{15a}$$

$$P_-^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(p) =\sum_m V_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\bar{V}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p}). \tag{15b}$$

Armed with the explicit expressions for $U_m^{\nu_1\nu_2\cdots\nu_n}$ and $V_m^{\nu_1\nu_2\cdots\nu_n}$, and the projection operators for spin 1 [see (5)] and 1/2 given by

$$\Lambda_+=\sum_{r=-1/2}^{1/2}u_r(\boldsymbol{p})\bar{u}_r(\boldsymbol{p})=\frac{\not p+\mathrm{i}W}{2\mathrm{i}E},\quad \Lambda_-=\sum_{r=-1/2}^{1/2}v_r(\boldsymbol{p})\bar{v}_r(\boldsymbol{p})=\frac{\not p-\mathrm{i}W}{2\mathrm{i}E}, \tag{16}$$

together with the following newly found sum relations concerning γ matrices and spin 1/2 and spin 1 wave functions:

$$\gamma_\nu e_{+1}^\nu u_{\frac{1}{2}}=0,\quad \frac{1}{\mathrm{i}\sqrt{2}}\gamma_5\gamma_\nu e_{+1}^\nu u_{-\frac{1}{2}}=u_{\frac{1}{2}},\quad \bar{u}_{\frac{1}{2}}\gamma_\nu\bar{e}_{+1}^\nu=0,\quad -\frac{1}{\mathrm{i}\sqrt{2}}\bar{u}_{-\frac{1}{2}}\gamma_\nu\bar{e}_{+1}^\nu\gamma_5=\bar{u}_{\frac{1}{2}}, \tag{17a}$$

$$\mathrm{i}\gamma_5\gamma_\nu e_0^\nu u_{\frac{1}{2}}=u_{\frac{1}{2}},\quad -\mathrm{i}\gamma_5\gamma_\nu e_0^\nu u_{-\frac{1}{2}}=u_{-\frac{1}{2}},\quad -\mathrm{i}\bar{u}_{\frac{1}{2}}\gamma_\nu e_0^\nu\gamma_5=\bar{u}_{\frac{1}{2}},\quad \mathrm{i}\bar{u}_{-\frac{1}{2}}\gamma_\nu e_0^\nu\gamma_5=\bar{u}_{-\frac{1}{2}}, \tag{17b}$$

$$-\frac{1}{\mathrm{i}\sqrt{2}}\gamma_5\gamma_\nu e_{-1}^\nu u_{\frac{1}{2}}=u_{-\frac{1}{2}},\quad \gamma_\nu e_{-1}^\nu u_{-\frac{1}{2}}=0,\quad \frac{1}{\mathrm{i}\sqrt{2}}\bar{u}_{\frac{1}{2}}\gamma_\nu\bar{e}_{-1}^\nu\gamma_5=\bar{u}_{-\frac{1}{2}},\quad \bar{u}_{-\frac{1}{2}}\gamma_\nu\bar{e}_{-1}^\nu=0, \tag{17c}$$

and

$$\gamma_5\Lambda_+\gamma_5=-\Lambda_-,\quad \gamma_\sigma P^{\sigma\mu}\Lambda_-=-\Lambda_+\gamma_\sigma P^{\sigma\mu},\quad \gamma_\sigma P^{\sigma\mu}\gamma_5\Lambda_+\gamma_5=\Lambda_+\gamma_\sigma P^{\sigma\mu}, \tag{18}$$

one can perform a direct calculation of the projection operator for an arbitrary half-integral spin. For example, in the case of spin 3/2, by utilizing the explicit form of the positive energy wave functions listed below:

$$U_{\frac{3}{2}}^\mu(\boldsymbol{p})=e_{+1}^\mu(\boldsymbol{p})u_{\frac{1}{2}}(\boldsymbol{p}),$$

$$U_{\frac{1}{2}}^\mu(\boldsymbol{p})=\sqrt{\frac{1}{3}}e_{+1}^\mu(\boldsymbol{p})u_{-\frac{1}{2}}(\boldsymbol{p})+\sqrt{\frac{2}{3}}e_0^\mu(\boldsymbol{p})u_{\frac{1}{2}}(\boldsymbol{p}),$$

$$U_{-\frac{1}{2}}^\mu(\boldsymbol{p})=\sqrt{\frac{2}{3}}e_0^\mu(\boldsymbol{p})u_{-\frac{1}{2}}(\boldsymbol{p})+\sqrt{\frac{1}{3}}e_{-1}^\mu(\boldsymbol{p})u_{\frac{1}{2}}(\boldsymbol{p}),$$

$$U_{-\frac{3}{2}}^\mu(\boldsymbol{p})=e_{-1}^\mu(\boldsymbol{p})u_{-\frac{1}{2}}(\boldsymbol{p}),$$

a straightforward calculation leads to

$$P_+^{\mu\nu}\left(\frac{3}{2},p\right)=\sum_{m=-3/2}^{3/2}U_m^\mu(\boldsymbol{p})\bar{U}_m^\nu(\boldsymbol{p}) \tag{19}$$

$$= \Lambda_+ \left[P^{\mu\nu}(p) - \frac{1}{3}\gamma_\sigma\gamma_\rho P^{\sigma\mu}(p)P^{\rho\nu}(p) \right].$$

Similarly,

$$P_-^{\mu\nu}\left(\frac{3}{2}, p\right) = \sum_{m=-3/2}^{3/2} V_m^\mu(\boldsymbol{p})\bar{V}_m^\nu(\boldsymbol{p}) \tag{20}$$
$$= \Lambda_- \left[P^{\mu\nu}(p) - \frac{1}{3}\gamma_\sigma\gamma_\rho P^{\sigma\mu}(p)P^{\rho\nu}(p) \right].$$

Combining (19) and (20), we get

$$P_\pm^{\mu\nu}\left(\frac{3}{2}, p\right) = \Lambda_\pm Q^{\mu\nu}\left(\frac{3}{2}, p\right), \tag{21a}$$
$$Q^{\mu\nu}\left(\frac{3}{2}, p\right) = P^{\mu\nu}(p) - \frac{1}{3}\gamma_\sigma\gamma_\rho P^{\sigma\mu}(p)P^{\rho\nu}(p). \tag{21b}$$

By using the following relations:

$$P^{\mu\nu} = P^{\nu\mu}, \quad \gamma_\sigma\gamma_\rho P^{\sigma\rho} = 3, \tag{21c}$$
$$\gamma_\sigma\gamma_\rho P^{\sigma\nu}P^{\rho\mu} = 2P^{\mu\nu} - \gamma_\sigma\gamma_\rho P^{\sigma\mu}P^{\rho\nu},$$

one can rewrite $Q^{\mu\nu}\left(\frac{3}{2}, p\right)$ as

$$Q^{\mu\nu}\left(\frac{3}{2}, p\right) = \frac{2}{5}\gamma_\sigma\gamma_\rho P^{\sigma\mu\rho\nu}(2, p), \tag{21d}$$

where $P^{\sigma\mu\rho\nu}(2, p)$ is the projection operator for spin 2. Similarly, in the case of spin 5/2, a direct calculation results in

$$P_\pm^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}, p\right) = \Lambda_\pm Q^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}, p\right),$$
$$Q^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}, p\right) = \frac{3}{7}\gamma_\sigma\gamma_\rho P^{\sigma\mu_1\mu_2\rho\nu_1\nu_2}(3, p), \tag{22}$$

where $P^{\sigma\mu_1\mu_2\rho\nu_1\nu_2}(3, p)$ is the projection operator for spin 3. From these results it is possible to write down a general expression for the spin $n + 1/2$ projection operator such as

$$P_\pm^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n + \frac{1}{2}, p\right) \tag{23a}$$
$$= \Lambda_\pm Q^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n + \frac{1}{2}, p\right),$$
$$Q^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n + \frac{1}{2}, p\right) \tag{23b}$$
$$= \frac{n+1}{2n+3}\gamma_\mu\gamma_\nu P^{\mu\mu_1\cdots\mu_n\nu\nu_1\cdots\nu_n}(n+1, p),$$

where $P^{\mu\mu_1\cdots\mu_n\nu\nu_1\cdots\nu_n}(n+1, p)$ is the projection operator for spin $n + 1$, namely

$$P^{\mu\mu_1\cdots\mu_n\nu\nu_1\cdots\nu_n}(n+1, p)$$
$$= \sum_{m=-(n+1)}^{n+1} e_m^{\mu\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\bar{e}_m^{\nu\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p}). \tag{24}$$

The expressions given by (23a) and (23b) are consistent with that constructed by Behrends and Fronsdal [7]. In what follows, we give a general proof for these expressions. Considering the positive energy projection operator, it can be written, using (11a) and (13a), as

$$P_+^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(n + 1/2, p)$$
$$= \sum_{m=-(n+1/2)}^{n+1/2} U_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\bar{U}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$$
$$= \frac{1}{2n+1}\sum_{m=-(n+1)}^{n+1} \Big[(n+1+m)e_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})u_{\frac{1}{2}}(\boldsymbol{p})$$
$$\times\bar{u}_{\frac{1}{2}}(\boldsymbol{p})\bar{e}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$$
$$+ \sqrt{(n+1+m)(n-m)}e_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})u_{\frac{1}{2}}(\boldsymbol{p})$$
$$\times\bar{u}_{-\frac{1}{2}}(\boldsymbol{p})\bar{e}_{m+1}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$$
$$+ \sqrt{(n+1+m)(n-m)}e_{m+1}^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})u_{-\frac{1}{2}}(\boldsymbol{p})$$
$$\times\bar{u}_{\frac{1}{2}}(\boldsymbol{p})\bar{e}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p}) \tag{25}$$
$$+ (n-m)e_{m+1}^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})u_{-\frac{1}{2}}(\boldsymbol{p})\ \bar{u}_{-\frac{1}{2}}(\boldsymbol{p})\bar{e}_{m+1}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p}) \Big]$$

(notice $e_m^{\mu_1\mu_2\cdots\mu_n} = 0$ when $m > n$ or $m < -n$). On the other hand, we have from (24) and (2)

$$\frac{n+1}{2n+3}\Lambda_+\gamma_\mu\gamma_\nu P^{\mu\mu_1\cdots\mu_n\nu\nu_1\cdots\nu_n}(n+1, p)$$
$$= \frac{n+1}{2n+3}\Lambda_+\gamma_\mu \sum_{m=-(n+1)}^{n+1}$$
$$\times \left[\sum_{\lambda=-1}^{1} \langle n, m-\lambda, 1, \lambda | n, 1, n+1, m\rangle\, e_{m-\lambda}^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p}) \right.$$
$$\left. \times e_\lambda^\mu(\boldsymbol{p}) \right]$$
$$\times \left[\sum_{\lambda=-1}^{1} \langle n, m-\lambda, 1, \lambda | n, 1, n+1, m\rangle\, \bar{e}_{m-\lambda}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p}) \right.$$
$$\left. \times\, \bar{e}_\lambda^\nu(\boldsymbol{p}) \right] \gamma_\nu$$
$$= \frac{\Lambda_+\gamma_\mu}{(2n+3)(2n+1)}$$
$$\times \sum_{m=-(n+1)}^{n+1} \left\{ \frac{(n+m+1)(n+m)}{2} \right.$$
$$\times e_{m-1}^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})e_1^\mu(\boldsymbol{p})\bar{e}_{m-1}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\bar{e}_1^\nu(\boldsymbol{p})$$
$$+ (n+m+1)\sqrt{\frac{(n+m)(n-m+1)}{2}}$$
$$\times e_{m-1}^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})e_1^\mu(\boldsymbol{p})\bar{e}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\bar{e}_0^\nu(\boldsymbol{p})$$
$$+ \frac{1}{2}\sqrt{(n^2-m^2)[(n+1)^2-m^2]}$$
$$\times e_{m-1}^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})e_1^\mu(\boldsymbol{p})\bar{e}_{m+1}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\bar{e}_{-1}^\nu(\boldsymbol{p})$$
$$+ (n+m+1)\sqrt{\frac{(n+m)(n-m+1)}{2}}$$
$$\times e_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})e_0^\mu(\boldsymbol{p})\bar{e}_{m-1}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\bar{e}_1^\nu(\boldsymbol{p})$$

$$
\begin{aligned}
&+[(n+1)^2-m^2]e_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})e_0^{\mu}(\boldsymbol{p})\bar{e}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\bar{e}_0^{\nu}(\boldsymbol{p})\\
&+(n-m+1)\sqrt{\frac{(n-m)(n+m+1)}{2}}\\
&\quad\times e_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})e_0^{\mu}(\boldsymbol{p})\bar{e}_{m+1}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\bar{e}_{-1}^{\nu}(\boldsymbol{p})\\
&+\frac{1}{2}\sqrt{(n^2-m^2)[(n+1)^2-m^2]}\\
&\quad\times e_{m+1}^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})e_{-1}^{\mu}(\boldsymbol{p})\bar{e}_{m-1}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\bar{e}_1^{\nu}(\boldsymbol{p})\\
&+(n-m+1)\sqrt{\frac{(n-m)(n+m+1)}{2}}\\
&\quad\times e_{m+1}^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})e_{-1}^{\mu}(\boldsymbol{p})\bar{e}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\bar{e}_0^{\nu}(\boldsymbol{p})\\
&+\frac{(n-m+1)(n-m)}{2}\\
&\quad\times e_{m+1}^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})e_{-1}^{\mu}(\boldsymbol{p})\bar{e}_{m+1}^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\bar{e}_{-1}^{\nu}(\boldsymbol{p})\Big\}\,\gamma_\nu,
\end{aligned}
\tag{26}
$$

where the Wigner formula for the Clebsch–Gordan coefficients has been used. Utilizing

$$
\begin{aligned}
&\Lambda_+\gamma_\mu e_\lambda^{\mu}(\boldsymbol{p})\\
&=\gamma_\mu e_\lambda^{\mu}(\boldsymbol{p})\gamma_5\Lambda_+\gamma_5=\gamma_\mu e_\lambda^{\mu}(\boldsymbol{p})\gamma_5\left(u_{\frac12}\bar{u}_{\frac12}+u_{-\frac12}\bar{u}_{-\frac12}\right)\gamma_5,
\end{aligned}
$$

and (17), it is not difficult to find that the right side of (26) is exactly the same as the right hand side of (25). Hence

$$
\begin{aligned}
&P_+^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac12,p\right)\\
&=\frac{n+1}{2n+3}\Lambda_+\gamma_\mu P^{\mu\mu_1\cdots\mu_n\nu\nu_1\cdots\nu_n}(n+1,p)\gamma_\nu.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&P_-^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac12,p\right)\\
&=\frac{n+1}{2n+3}\Lambda_-\gamma_\mu P^{\mu\mu_1\cdots\mu_n\nu\nu_1\cdots\nu_n}(n+1,p)\gamma_\nu.
\end{aligned}
$$

Therefore, the B-F form of the projection operator for an arbitrary half-integral spin is proved. In view of the calculation of the propagator, however, the B-F form is not the simplest one because it contains many non-independent terms. Fortunately, this problem can be solved by the following tricks. On one hand, by using the explicit expression for the spin $n+1$ projection operator and (21d), one can rewrite $Q^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac12,p\right)$ in a form that contains only the independent terms, namely, $3P^{\mu_1\nu_1}P^{\mu_2\nu_2}\dots P^{\mu_n\nu_n}$, $\gamma_\sigma\gamma_\rho P^{\sigma\mu_1}P^{\rho\nu_1}P^{\mu_2\nu_2}P^{\mu_n\nu_n}$ and their permutations among the tensor indexes $(\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n)$. On the other hand, by using

$$
\begin{aligned}
P^{\mu\nu}&=\delta_{\mu\nu}+\frac{p_\mu p_\nu}{W^2},\\
p^2&=-W^2,\\
\Lambda_\pm\,\not{p}&=\pm\mathrm{i}W\Lambda_\pm,
\end{aligned}
\tag{27}
$$

one can carry out the contraction in terms like $\Lambda_\pm\gamma_\sigma\gamma_\rho P^{\sigma\mu_1}P^{\rho\nu_1}P^{\mu_2\nu_2}P^{\mu_n\nu_n}$ to find that

$$
\begin{aligned}
&\Lambda_\pm\gamma_\sigma\gamma_\rho P^{\sigma\mu_1}P^{\rho\nu_1}P^{\mu_2\nu_2}\dots P^{\mu_n\nu_n}\\
&=\Lambda_\pm\left[\gamma_{\mu_1}\gamma_{\nu_1}\mp\frac{\mathrm{i}}{W}\left(\gamma_{\mu_1}p_{\nu_1}-\gamma_{\nu_1}p_{\mu_1}\right)+\frac{1}{W^2}p_{\mu_1}p_{\nu_1}\right]\\
&\quad\times P^{\mu_2\nu_2}\dots P^{\mu_n\nu_n}.
\end{aligned}
\tag{28}
$$

After these simplifications, the projection operators can be expressed in a form that is suitable for the calculation of the propagators. For example, in the cases of the spins 3/2, 5/2 and 7/2, the projection operators become respectively

$$
\begin{aligned}
&P_\pm^{\mu\nu}\left(\frac32,p\right)\\
&=\Lambda_\pm\left[\delta_{\mu\nu}-\frac13\gamma_\mu\gamma_\nu\pm\frac{\mathrm{i}}{3W}\left(\gamma_\mu p_\nu-\gamma_\nu p_\mu\right)\right.\\
&\qquad\left.+\frac{2}{3W^2}p_\mu p_\nu\right],
\end{aligned}
\tag{29a}
$$

$$
\begin{aligned}
&P_\pm^{\mu_1\mu_2\nu_1\nu_2}\left(\frac52,p\right)\\
&=\Lambda_\pm\left\{\frac12\left(P^{\mu_1\nu_1}P^{\mu_2\nu_2}+P^{\mu_1\nu_2}P^{\mu_2\nu_1}\right)-\frac15P^{\mu_1\mu_2}P^{\nu_1\nu_2}\right.\\
&\quad-\frac{1}{10}\sum_{\substack{P(\mu_1\mu_2)\\P(\nu_1\nu_2)}}\left[\gamma_{\mu_1}\gamma_{\nu_1}\mp\frac{\mathrm{i}}{W}\left(\gamma_{\mu_1}p_{\nu_1}-\gamma_{\nu_1}p_{\mu_1}\right)\right.\\
&\qquad\left.\left.+\frac{1}{W^2}p_{\mu_1}p_{\nu_1}\right]P^{\mu_2\nu_2}\right\},
\end{aligned}
\tag{29b}
$$

$$
\begin{aligned}
&P_\pm^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\left(\frac72,p\right)\\
&=\Lambda_\pm\left\{\frac16\sum_{P(\nu_1\nu_2\nu_3)}P^{\mu_1\nu_1}P^{\mu_2\nu_2}P^{\mu_3\nu_3}\right.\\
&\quad-\frac{1}{280}\sum_{\substack{P(\mu_1\mu_2\mu_3)\\P(\nu_1\nu_2\nu_3)}}P^{\mu_1\mu_2}P^{\nu_1\nu_2}P^{\mu_3\nu_3}\\
&\quad-\frac{1}{84}\sum_{\substack{P(\mu_1\mu_2\mu_3)\\P(\nu_1\nu_2\nu_3)}}\left[\left(\gamma_{\mu_1}\gamma_{\nu_1}\mp\frac{\mathrm{i}}{W}\left(\gamma_{\mu_1}p_{\nu_1}-\gamma_{\nu_1}p_{\mu_1}\right)\right.\right.\\
&\qquad\left.+\frac{1}{W^2}p_{\mu_1}p_{\nu_1}\right)P^{\mu_2\nu_2}P^{\mu_3\nu_3}\\
&\quad+\frac29\left(\gamma_{\mu_1}\gamma_{\mu_2}\mp\frac{\mathrm{i}}{W}\left(\gamma_{\mu_1}p_{\mu_2}-\gamma_{\mu_2}p_{\mu_1}\right)\right.\\
&\qquad\left.+\frac{1}{W^2}p_{\mu_1}p_{\mu_2}\right)P^{\nu_1\nu_2}P^{\mu_3\nu_3}\\
&\quad+\frac15\left(\gamma_{\nu_1}\gamma_{\mu_1}\mp\frac{\mathrm{i}}{W}\left(\gamma_{\nu_1}p_{\mu_1}-\gamma_{\mu_1}p_{\nu_1}\right)\right.\\
&\qquad\left.\left.\left.+\frac{1}{W^2}p_{\mu_1}p_{\nu_1}\right)P^{\mu_2\mu_3}P^{\nu_2\nu_3}\right]\right\},
\end{aligned}
\tag{29c}
$$

where $P(\mu_1\mu_2\cdots\mu_n)$ stands for permutations of $(\mu_1,\mu_2,\cdots,\mu_n)$.

3 The Feynman propagator

3.1 Integral spin

We now proceed to derive the Feynman propagator for an arbitrary integral spin. We begin by working out the Feynman propagator for spin 2 in a procedure similar to the one by which the propagator for spin 1 [13] is derived.

3.1.1 Spin 2

The wave functions for spin 2 can be expressed as

$$A^{\nu_1\nu_2}(x) = \sum_{\boldsymbol{p}} \sum_{m=-2}^{2} \frac{1}{\sqrt{2EV}} \times \left[a_m(\boldsymbol{p}) e_m^{\nu_1\nu_2}(\boldsymbol{p}) \mathrm{e}^{\mathrm{i}px} + b_m^+(\boldsymbol{p}) \bar{e}_m^{\nu_1\nu_2}(\boldsymbol{p}) \mathrm{e}^{-\mathrm{i}px}\right] = A^{\nu_1\nu_2}(x)^{(-)} + A^{\nu_1\nu_2}(x)^{(+)}, \tag{30}$$

where the superscripts (+) and (−) denote creation and destruction parts respectively, and

$$e_m^{\nu_1\nu_2}(\boldsymbol{p}) = \sum_{\lambda_1\lambda_2} e_{\lambda_1}^{\nu_1}(\boldsymbol{p}) e_{\lambda_2}^{\nu_2}(\boldsymbol{p}) \langle 1,\lambda_1;1,\lambda_2|1,1,2,m\rangle, \tag{31a}$$

$$\bar{e}_m^{\nu_1\nu_2}(\boldsymbol{p}) = \sum_{\lambda_1\lambda_2} \bar{e}_{\lambda_1}^{\nu_1}(\boldsymbol{p}) \bar{e}_{\lambda_2}^{\nu_2}(\boldsymbol{p}) \langle 1,\lambda_1;1,\lambda_2|1,1,2,m\rangle, \tag{31b}$$

and they are normalized according to

$$\bar{e}_m^{\nu_1\nu_2}(\boldsymbol{p}) e_{m'}^{\nu_1\nu_2}(\boldsymbol{p}) = \delta_{m,m'}, \quad m,m' = 2,1,0,-1,-2. \tag{32}$$

Utilizing the projection operator given by (6), the field quantization conditions

$$\left[a_m(\boldsymbol{p}), a_{m'}^+(\boldsymbol{p}')\right] = \delta_{\boldsymbol{p},\boldsymbol{p}'}\delta_{mm'} = \left[b_m(\boldsymbol{p}), b_{m'}^+(\boldsymbol{p}')\right], \tag{33}$$

with all others vanishing, and the definition of the adjoint field

$$\bar{A}^{\nu_1\nu_2}(x) = g_{\nu_1\mu_1} g_{\nu_1\mu_2} \left(A_m^{\mu_1\mu_2}(\boldsymbol{p})\right)^+ = \sum_{\boldsymbol{p},m} \frac{1}{\sqrt{2EV}} \left[a_m^+(\boldsymbol{p}) \bar{e}_m^{\nu_1\nu_2}(\boldsymbol{p}) \mathrm{e}^{-\mathrm{i}px} + b_m^(\boldsymbol{p}) e_m^{\nu_1\nu_2}(\boldsymbol{p}) \mathrm{e}^{\mathrm{i}px}\right] = \bar{A}^{\nu_1\nu_2}(x)^{(+)} + \bar{A}^{\nu_1\nu_2}(x)^{(-)}, \tag{34}$$

a general commutation rule could be derived and can be expressed as

$$\left[A^{\mu_1\mu_2}(x), \bar{A}^{\nu_1\nu_2}(x')\right] = \mathrm{i}\hat{P}^{\mu_1\mu_2\nu_1\nu_2}(2)\Delta(x-x'), \tag{35}$$

or equivalently

$$\left[A^{\mu_1\mu_2}(x)^{(-)}, \bar{A}^{\nu_1\nu_2}(x')^{(+)}\right] = \mathrm{i}\hat{P}^{\mu_1\mu_2\nu_1\nu_2}(2)\Delta^{(+)}(x-x'), \tag{36a}$$

$$\left[A^{\mu_1\mu_2}(x)^{(+)}, \bar{A}^{\nu_1\nu_2}(x')^{(-)}\right] = \mathrm{i}\hat{P}^{\mu_1\mu_2\nu_1\nu_2}(2)\Delta^{(-)}(x-x'), \tag{36b}$$

where

$$\mathrm{i}\Delta(x-x') = \mathrm{i}\Delta^{(+)}(x-x') + \mathrm{i}\Delta^{(-)}(x-x'), \tag{37a}$$

$$\mathrm{i}\Delta^{(+)}(x-x') = \sum_{(\boldsymbol{p})} \frac{1}{2EV} \mathrm{e}^{\mathrm{i}p(x-x')},$$

$$\mathrm{i}\Delta^{(-)}(x-x') = -\sum_{(\boldsymbol{p})} \frac{1}{2\omega V} \mathrm{e}^{-\mathrm{i}p(x-x')}, \tag{37b}$$

and

$$\hat{P}^{\mu_1\mu_2\nu_1\nu_2}(2) = \frac{1}{4}\sum_{\substack{P(\mu)\\P(\nu)}} \left[\hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2} - \frac{1}{3}\hat{P}^{\mu_1\mu_2}\hat{P}^{\nu_1\nu_2}\right],$$

$$\hat{P}^{\mu\nu} = \delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2}. \tag{38}$$

The Feynman propagator for spin 2 is defined in a similar manner as in the case of spin 1 [13] according to

$$\Delta_{\mathrm{F}}^{\mu_1\mu_2\nu_1\nu_2}(x-x') \equiv \langle 0|\, T A^{\mu_1\mu_2}(x)\bar{A}^{\nu_1\nu_2}(x')|0\rangle = \begin{cases} \langle 0|\, A^{\mu_1\mu_2}(x)\bar{A}^{\nu_1\nu_2}(x')|0\rangle & t>t', \\ \langle 0|\, \bar{A}^{\nu_1\nu_2}(x')A^{\mu_1\mu_2}(x)|0\rangle & t<t'. \end{cases} \tag{39}$$

With the help of (36), it is not difficult to find

$$\Delta_{\mathrm{F}}^{\mu_1\mu_2\nu_1\nu_2}(x-x') = \mathrm{i}\theta(t-t')\hat{P}^{\mu_1\mu_2\nu_1\nu_2}\Delta^{(+)}(x-x') - \mathrm{i}\theta(t'-t)\hat{P}^{\mu_1\mu_2\nu_1\nu_2}\Delta^{(-)}(x-x'), \tag{40a}$$

or (setting $x'=0$)

$$\Delta_{\mathrm{F}}^{\mu_1\mu_2\nu_1\nu_2}(x) = \mathrm{i}\theta(t)\hat{P}^{\mu_1\mu_2\nu_1\nu_2}\Delta^{(+)}(x) - \mathrm{i}\theta(-t)\hat{P}^{\mu_1\mu_2\nu_1\nu_2}\Delta^{(-)}(x). \tag{40b}$$

By virtue of

$$\partial_t\theta(t) = \delta(t), \quad \delta(t)\left[\Delta^{(+)}(x) + \Delta^{(-)}(x)\right] = 0,$$

$$\dot{\delta}(t)\left[\Delta^{(+)}(x) + \Delta^{(-)}(x)\right] = \delta^{(4)}(x), \tag{41}$$

performing a calculation that makes the differential operator $\hat{P}^{\mu_1\mu_2\nu_1\nu_2}(2)$ commute past the θ functions, it is found that

$$\mathrm{i}\left[\theta(t)\hat{P}^{\mu_1\nu_1}\Delta^{(+)}(x) - \theta(-t)\hat{P}^{\mu_1\nu_1}\Delta^{(-)}(x)\right] = \hat{P}^{\mu_1\nu_1}\Delta_{\mathrm{F}}(x) + \frac{\mathrm{i}}{W^2}\delta_{\mu_1 4}\delta_{\nu_1 4}\delta^{(4)}(x), \tag{42a}$$

and

$$\mathrm{i}\left[\theta(t)\hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}\Delta^{(+)}(x) - \theta(-t)\hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}\Delta^{(-)}(x)\right] = \hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}\Delta_{\mathrm{F}}(x) + \frac{\mathrm{i}}{W^2}\left(\delta_{\mu_1\nu_1}\delta_{\mu_2 4}\delta_{\nu_2 4} + \delta_{\mu_1 4}\delta_{\nu_1 4}\hat{P}^{\mu_2\nu_2}\right)\delta^{(4)}(x), \tag{42b}$$

with $\Delta_{\mathrm{F}}(x) = \mathrm{i}\theta(t)\Delta^{(+)}(x) - \mathrm{i}\theta(-t)\Delta^{(-)}(x)$; hence $\Delta_{\mathrm{F}}^{\mu_1\mu_2\nu_1\nu_2}(x-x')$ can be rewritten as

$$\Delta_{\mathrm{F}}^{\mu_1\mu_2\nu_1\nu_2}(x-x') = \hat{P}^{\mu_1\mu_2\nu_1\nu_2}(2)\Delta_{\mathrm{F}}(x-x') + \hat{K}^{\mu_1\mu_2\nu_1\nu_2}(2)\delta^{(4)}(x-x'), \tag{43}$$

where

$$\Delta_{\mathrm{F}}(x-x') = \mathrm{i}\theta(t-t')\Delta^{(+)}(x-x') - \mathrm{i}\theta(t'-t)\Delta^{(-)}(x-x'), \tag{44a}$$

$$\hat{K}^{\mu_1\mu_2\nu_1\nu_2}(2) = \frac{\mathrm{i}}{4W^2}\sum_{\substack{P(\mu)\\P(\nu)}}\Big[\delta_{\mu_1\nu_1}\delta_{\mu_2 4}\delta_{\nu_2 4} + \delta_{\mu_1 4}\delta_{\nu_1 4}\hat{P}^{\mu_2\nu_2} - \frac{1}{3}\left(\delta_{\nu_1 4}\delta_{\nu_2 4}\delta_{\mu_1\mu_2} + \delta_{\mu_1 4}\delta_{\mu_2 4}\hat{P}^{\nu_1\nu_2}\right)\Big]. \tag{44b}$$

This is the Feynman propagator for spin 2 in coordinate representation. Using

$$\Delta_{\mathrm{F}}(x) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4p\,\mathrm{e}^{\mathrm{i}px}\Delta_{\mathrm{F}}(p), \qquad \Delta_{\mathrm{F}}(p) = \frac{-\mathrm{i}}{p^2+W^2-\mathrm{i}\varepsilon}, \tag{45}$$

the Fourier representation for $\Delta_{\mathrm{F}}^{\mu_1\mu_2\nu_1\nu_2}(x)$ can be easily deduced

$$\Delta_{\mathrm{F}}^{\mu_1\mu_2\nu_1\nu_2}(x) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4p\,\mathrm{e}^{\mathrm{i}px}\Delta_{\mathrm{F}}^{\mu_1\mu_2\nu_1\nu_2}(p), \tag{46a}$$

$$\Delta_{\mathrm{F}}^{\mu_1\mu_2\nu_1\nu_2}(p) = P^{\mu_1\mu_2\nu_1\nu_2}(2,p)\frac{-\mathrm{i}}{p^2+W^2-\mathrm{i}\varepsilon} + K^{\mu_1\mu_2\nu_1\nu_2}(2,p), \tag{46b}$$

where

$$K^{\mu_1\mu_2\nu_1\nu_2}(2,p) = \frac{\mathrm{i}}{4W^2}\sum_{\substack{P(\mu)\\P(\nu)}}[\delta_{\mu_1\nu_1}\delta_{\mu_2 4}\delta_{\nu_2 4} + \delta_{\mu_1 4}\delta_{\nu_1 4}P^{\mu_2\nu_2} - \frac{1}{3}(\delta_{\nu_1 4}\delta_{\nu_2 4}\delta_{\mu_1\mu_2} + \delta_{\mu_1 4}\delta_{\mu_2 4}P^{\nu_1\nu_2})]. \tag{47}$$

Equation (46b) gives the Feynman propagator for spin 2 in momentum representation. We emphasize here that the second part on the right hand side of (43) or (46b) represents the extra non-covariant term that inevitably appears in the expression of the propagator for spin larger than 1/2.

3.1.2 Spin 3

The wave functions for spin 3 are

$$A^{\nu_1\nu_2\nu_3}(x) = \sum_{\boldsymbol{p},m}\frac{1}{\sqrt{2EV}}\left[a_m(\boldsymbol{p})e_m^{\nu_1\nu_2\nu_3}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px} + b_m^+(\boldsymbol{p})\bar{e}_m^{\nu_1\nu_2\nu_3}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}\right] = A^{\nu_1\nu_2\nu_3}(x)^{(-)} + A^{\nu_1\nu_2\nu_3}(x)^{(+)}, \tag{48}$$

where

$$e_m^{\nu_1\nu_2\nu_3}(\boldsymbol{p}) = \sum_{\lambda_{12}\lambda_3} e_{\lambda_{12}}^{\nu_1\nu_2}(\boldsymbol{p})e_{\lambda_3}^{\nu_3}(\boldsymbol{p})\langle 2,\lambda_{12};1,\lambda_3|2,1,3,m\rangle, \tag{49a}$$

$$\bar{e}_m^{\nu_1\nu_2\nu_3}(\boldsymbol{p}) = \sum_{\lambda_{12}\lambda_3} \bar{e}_{\lambda_{12}}^{\nu_1\nu_2}(\boldsymbol{p})\bar{e}_{\lambda_3}^{\nu_3}(\boldsymbol{p})\langle 2,\lambda_{12};1,\lambda_3|2,1,3,m\rangle, \tag{49b}$$

and they are normalized according to

$$\bar{e}_m^{\nu_1\nu_2\nu_3}(\boldsymbol{p})e_{m'}^{\nu_1\nu_2\nu_3}(\boldsymbol{p}) = \delta_{m,m'}, \quad m,m' = 3,2,1,0,-1,-2,-3. \tag{50}$$

As in the case of spin 2, by using (7), (33) and the definition of the adjoint field

$$\bar{A}^{\nu_1\nu_2\nu_3}(x) = g_{\nu_1\mu_1}g_{\nu_1\mu_2}g_{\nu_3\mu_3}\left(A_m^{\mu_1\mu_2\mu_3}(\boldsymbol{p})\right)^+ = \sum_{\boldsymbol{p},m}\frac{1}{\sqrt{2EV}} \times[a_m^+(\boldsymbol{p})\bar{e}_m^{\nu_1\nu_2\nu_3}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px} + b_m(\boldsymbol{p})e_m^{\nu_1\nu_2\nu_3}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}] = \bar{A}^{\nu_1\nu_2\nu_3}(x)^{(+)} + \bar{A}^{\nu_1\nu_2\nu_3}(x)^{(-)}, \tag{51}$$

we find that the general commutation relation takes the following form:

$$\left[A^{\mu_1\mu_2\mu_3}(x), \bar{A}^{\nu_1\nu_2\nu_3}(x')\right] = \mathrm{i}\hat{P}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(3)\Delta(x-x'), \tag{52}$$

or equivalently

$$\left[A^{\mu_1\mu_2\mu_3}(x)^{(-)}, \bar{A}^{\nu_1\nu_2\nu_3}(x')^{(+)}\right] = \mathrm{i}\hat{P}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(3)\Delta^{(+)}(x-x'), \tag{53a}$$

$$\left[A^{\mu_1\mu_2\mu_3}(x)^{(+)}, \bar{A}^{\nu_1\nu_2\nu_3}(x')^{(-)}\right] = \mathrm{i}\hat{P}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(3)\Delta^{(-)}(x-x'), \tag{53b}$$

where

$$\hat{P}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(3) = \frac{1}{36}\sum_{\substack{P(\mu_1\mu_2\mu_3)\\P(\nu_1\nu_2\nu_3)}}\left[\hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}\hat{P}^{\mu_3\nu_3} - \frac{3}{5}\hat{P}^{\mu_1\mu_2}\hat{P}^{\nu_1\nu_2}\hat{P}^{\mu_3\nu_3}\right]. \tag{54}$$

The Feynman propagator for spin 3 is defined by

$$\Delta_{\mathrm{F}}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(x-x') \equiv \langle 0|\,TA^{\mu_1\mu_2\mu_3}(x)\bar{A}^{\nu_1\nu_2\nu_3}(x')|0\rangle = \begin{cases}\langle 0|\,A^{\mu_1\mu_2\mu_3}(x)\bar{A}^{\nu_1\nu_2\nu_3}(x')|0\rangle & t>t',\\ \langle 0|\,\bar{A}^{\nu_1\nu_2\nu_3}(x')A^{\mu_1\mu_2\mu_3}(x)|0\rangle & t<t'.\end{cases} \tag{55}$$

It follows from (53) that

$$\Delta_{\mathrm{F}}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(x-x')$$

$$= \mathrm{i}\theta(t-t')\hat{P}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(3)\Delta^{(+)}(x-x') \\ -\mathrm{i}\theta(t'-t)\hat{P}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(3)\Delta^{(-)}(x-x'). \tag{56}$$

Based on (42a) and (42b), we further find

$$\Big[\mathrm{i}\theta(t)\hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}\hat{P}^{\mu_3\nu_3}\Delta^{(+)}(x) \\ -\mathrm{i}\theta(-t)\hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}\hat{P}^{\mu_3\nu_3}\Delta^{(-)}(x)\Big] \tag{57}$$
$$= \hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}\hat{P}^{\mu_3\nu_3}\Delta_{\mathrm{F}}(x) \\ +\frac{\mathrm{i}}{W^2}\left[\left(\delta_{\mu_1\nu_1}\delta_{\mu_2 4}\delta_{\nu_2 4}+\delta_{\mu_1 4}\delta_{\nu_1 4}\hat{P}^{\mu_2\nu_2}\right)\hat{P}^{\mu_3\nu_3} \\ +\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2}\delta_{\mu_3 4}\delta_{\nu_3 4}\right]\delta^{(4)}(x),$$

thus the Feynman propagator for spin 3 in coordinate representation can be expressed as

$$\Delta_{\mathrm{F}}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(x-x') \\ = \hat{P}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(3)\Delta_{\mathrm{F}}(x-x') \\ +\hat{K}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(3)\delta^{(4)}(x-x'), \tag{58}$$

where

$$\hat{K}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(3) \\ = \frac{\mathrm{i}}{36W^2}\sum_{\substack{P(\mu)\\P(\nu)}}\Big\{\left(\delta_{\mu_1\nu_1}\delta_{\mu_2 4}\delta_{\nu_2 4}+\delta_{\mu_1 4}\delta_{\nu_1 4}\hat{P}^{\mu_2\nu_2}\right)\hat{P}^{\mu_3\nu_3} \\ +\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2}\delta_{\mu_3 4}\delta_{\nu_3 4} \\ -\frac{3}{5}\left[\left(\delta_{\mu_1\mu_2}\delta_{\nu_1 4}\delta_{\nu_2 4}+\delta_{\mu_1 4}\delta_{\mu_2 4}\hat{P}^{\nu_1\nu_2}\right)\hat{P}^{\mu_3\nu_3} \\ +\delta_{\mu_1\mu_2}\delta_{\nu_1\nu_2}\delta_{\mu_3 4}\delta_{\nu_3 4}\right]\Big\}. \tag{59}$$

The Fourier representation for $\Delta_{\mathrm{F}}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(x)$ is, using (45), given by

$$\Delta_{\mathrm{F}}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(x) \\ = \frac{1}{(2\pi)^4}\int \mathrm{d}^4 p\mathrm{e}^{\mathrm{i}px}\Delta_{\mathrm{F}}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(p), \tag{60}$$
$$\Delta_{\mathrm{F}}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(p) \\ = P^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(3,p)\frac{-\mathrm{i}}{p^2+W^2-\mathrm{i}\varepsilon} \\ +K^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(3,p), \tag{61}$$

with

$$K^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(3) \tag{62}$$
$$= \frac{\mathrm{i}}{36W^2}\sum_{\substack{P(\mu)\\P(\nu)}}\Bigg\{\left(\delta_{\mu_1\nu_1}\delta_{\mu_2 4}\delta_{\nu_2 4}+\delta_{\mu_1 4}\delta_{\nu_1 4}P^{\mu_2\nu_2}\right)P^{\mu_3\nu_3} \\ +\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2}\delta_{\mu_3 4}\delta_{\nu_3 4} \\ -\frac{3}{5}\Bigg[\left(\delta_{\mu_1\mu_2}\delta_{\nu_1 4}\delta_{\nu_2 4}+\delta_{\mu_1 4}\delta_{\mu_2 4}P^{\nu_1\nu_2}\right)P^{\mu_3\nu_3} \\ +\delta_{\mu_1\mu_2}\delta_{\nu_1\nu_2}\delta_{\mu_3 4}\delta_{\nu_3 4}\Bigg]\Bigg\}.$$

Equation (61b) gives the Feynman propagator for spin 3 in momentum representation.

3.1.3 Spin n

The above procedure is now extended to the general case of an arbitrary integral spin n. We begin by working out the general commutation rules. From (1a), (8) and (33) and the definition of the adjoint field

$$\bar{A}^{\nu_1\nu_2\cdots\nu_n}(x) \\ = g_{\nu_1\mu_1}g_{\nu_1\mu_2}\cdots g_{\nu_n\mu_n}\left(A_m^{\mu_1\mu_2\cdots\mu_n}(\boldsymbol{p})\right)^+ \\ = \sum_{\boldsymbol{p},m}\frac{1}{\sqrt{2EV}}\Big[a_m^+(\boldsymbol{p})\bar{e}_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px} \\ +b_m(\boldsymbol{p})e_m^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}\Big] \\ = \bar{A}^{\nu_1\nu_2\cdots\nu_n}(x)^{(+)}+\bar{A}^{\nu_1\nu_2\cdots\nu_n}(x)^{(-)}, \tag{63}$$

we get, by following the same steps as in the cases of spin 2 and 3

$$\left[A^{\mu_1\mu_2\cdots\mu_n}(x),\bar{A}^{\nu_1\nu_2\cdots\nu_n}(x')\right] \\ = \mathrm{i}\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(n)\Delta(x-x'), \tag{64}$$

or

$$\left[A^{\mu_1\mu_2\cdots\mu_n}(x)^{(-)},\bar{A}^{\nu_1\nu_2\cdots\nu_n}(x')^{(+)}\right] \\ = \mathrm{i}\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(n)\Delta^{(+)}(x-x'), \tag{65a}$$
$$\left[A^{\mu_1\mu_2\cdots\mu_n}(x)^{(+)},\bar{A}^{\nu_1\nu_2\cdots\nu_n}(x')^{(-)}\right] \\ = \mathrm{i}\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(n)\Delta^{(-)}(x-x'), \tag{65b}$$

where

$$\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(n) \tag{66}$$
$$= \left(\frac{1}{n!}\right)^2\sum_{\substack{P(\mu)\\P(\nu)}}\left[\prod_{i=1}^{n}\hat{P}^{\mu_i\nu_i}+A_1(n)\hat{P}^{\mu_1\mu_2}\hat{P}^{\nu_1\nu_2}\prod_{i=3}^{n}\hat{P}^{\mu_i\nu_i} \\ +\cdots+\begin{cases}A_{n/2}(n)\hat{P}^{\mu_1\mu_2}\hat{P}^{\nu_1\nu_2}\ldots\hat{P}^{\mu_{n-1}\mu_n}\hat{P}^{\nu_{n-1}\nu_n}\\ \qquad(\text{for even } n)\\ A_{(n-1)/2}(n)\hat{P}^{\mu_1\mu_2}\hat{P}^{\nu_1\nu_2}\\ \quad\ldots\hat{P}^{\mu_{n-2}\mu_{n-1}}\hat{P}^{\nu_{n-2}\nu_{n-1}}\hat{P}^{\mu_n\nu_n}\\ \qquad(\text{for odd } n)\end{cases}\right].$$

The Feynman propagator for spin n is defined by

$$\Delta_{\mathrm{F}}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(x-x') \tag{67}$$
$$\equiv \langle 0|\,TA^{\mu_1\mu_2\cdots\mu_n}(x)\bar{A}^{\nu_1\nu_2\cdots\nu_n}(x')|0\rangle \\ = \begin{cases}\langle 0|\,A^{\mu_1\mu_2\cdots\mu_n}(x)\bar{A}^{\nu_1\nu_2\cdots\nu_n}(x')|0\rangle & t>t',\\ \langle 0|\,\bar{A}^{\nu_1\nu_2\cdots\nu_n}(x')A^{\mu_1\mu_2\cdots\mu_n}(x)|0\rangle & t<t',\end{cases}$$

and can be rewritten, using (65),

$$\Delta_{\mathrm{F}}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(x-x') \\ = \mathrm{i}\theta(t-t')\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(n)\Delta^{(+)}(x-x') \tag{68}$$

$$-\mathrm{i}\theta(t'-t)\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(n)\Delta^{(-)}(x-x').$$

or, as in the cases of spin 2 and 3,

$$\begin{aligned}&\Delta_{\mathrm{F}}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(x-x')\\&=\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(n)\Delta_{\mathrm{F}}(x-x')\\&\quad+\hat{K}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(n)\delta^{(4)}(x-x'),\end{aligned}\tag{69}$$

where the second term denotes the extra non-covariant term, which arises when the θ functions are commuted past the differential operator $\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(n)$. Based on (42a), (42b) and (57), after a step-by-step calculation, it is found that

$$\begin{aligned}&\Big[\mathrm{i}\theta(t)\hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}\ldots\hat{P}^{\mu_n\nu_n}\Delta^{(+)}(x)\\&\quad-\mathrm{i}\theta(-t)\hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}\ldots\hat{P}^{\mu_n\nu_n}\Delta^{(-)}(x)\Big]\\&=\hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}\ldots\hat{P}^{\mu_n\nu_n}\Delta_{\mathrm{F}}(x)\\&\quad+\frac{\mathrm{i}}{W^2}\left[\hat{B}^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}\right]\delta^{(4)}(x),\end{aligned}\tag{70}$$

where

$$\begin{aligned}\hat{B}^{\mu_1\nu_1}&=[\delta_{\mu_1 4}\delta_{\nu_1 4}],\\\hat{B}^{\mu_1\nu_1\mu_2\nu_2}&=\hat{B}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}+\delta_{\mu_1\nu_1}\delta_{\mu_2 4}\delta_{\nu_2 4},\\\hat{B}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}&=\hat{B}^{\mu_1\nu_1\mu_2\nu_2}\hat{P}^{\mu_3\nu_3}\\&\quad+\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2}\delta_{\mu_3 4}\delta_{\nu_3 4},\cdots,\\\hat{B}^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}&=\hat{B}^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_{n-1}\nu_{n-1}}\hat{P}^{\mu_n\nu_n}\\&\quad+\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2}\cdots\delta_{\mu_{n-1}\nu_{n-1}}\delta_{\mu_n 4}\delta_{\nu_n 4}.\end{aligned}\tag{71}$$

Therefore

$$\begin{aligned}&\hat{K}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(n)\\&=\frac{\mathrm{i}}{(n!W)^2}\sum_{\substack{P(\mu)\\P(\nu)}}\Bigg[\hat{B}^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}\\&\quad+A_1(n)\hat{B}^{\mu_1\mu_2\nu_1\nu_2\mu_3\nu_3\mu_4\nu_4\cdots\mu_n\nu_n}\\&\quad+\cdots+\begin{cases}A_{n/2}(n)\hat{B}^{\mu_1\mu_2\nu_1\nu_2\cdots\mu_{n-1}\mu_n\nu_{n-1}\nu_n}\\\quad\text{(for even } n)\\A_{(n-1)/2}(n)\hat{B}^{\mu_1\mu_2\nu_1\nu_2\cdots\mu_{n-2}\mu_{n-1}\nu_{n-2}\nu_{n-1}\mu_n\nu_n}\\\quad\text{(for odd } n)\end{cases}\Bigg],\end{aligned}\tag{72}$$

and (69) serves as a general expression for the Feynman propagator of spin n in coordinate representation. The Fourier representation for $\Delta_{\mathrm{F}}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu}(x)$ can be easily derived, using (45),

$$\Delta_{\mathrm{F}}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(x)=\frac{1}{(2\pi)^4}\int\mathrm{d}^4p\,\mathrm{e}^{\mathrm{i}px}\Delta_{\mathrm{F}}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(n,p),\tag{73a}$$

$$\begin{aligned}&\Delta_{\mathrm{F}}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(n,p)\\&=P^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(n,p)\frac{-\mathrm{i}}{p^2+W^2-\mathrm{i}\varepsilon}\\&\quad+K^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(n,p),\end{aligned}\tag{73b}$$

where

$$\begin{aligned}&K^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(n,k)\\&=\frac{\mathrm{i}}{(n!W)^2}\sum_{\substack{P(\mu_1\mu_2\cdots\mu_n)\\P(\nu_1\nu_2\cdots\nu_n)}}\Bigg[B^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}\\&\quad+A_1(n)B^{\mu_1\mu_2\nu_1\nu_2\mu_3\nu_3\mu_4\nu_4\cdots\mu_n\nu_n}\\&\quad+\cdots+\begin{cases}A_{n/2}(n)B^{\mu_1\mu_2\nu_1\nu_2\cdots\mu_{n-1}\mu_n\nu_{n-1}\nu_n}\\\quad\text{(for even } n)\\A_{(n-1)/2}(n)B^{\mu_1\mu_2\nu_1\nu_2\cdots\mu_{n-2}\mu_{n-1}\nu_{n-2}\nu_{n-1}\mu_n\nu_n}\\\quad\text{(for odd } n)\end{cases}\Bigg],\end{aligned}\tag{74}$$

with

$$\begin{aligned}B^{\mu_1\nu_1}&=\delta_{\mu_1 4}\delta_{\nu_1 4},\\B^{\mu_1\nu_1\mu_2\nu_2}&=B^{\mu_1\nu_1}P^{\mu_2\nu_2}+\delta_{\mu_1\nu_1}\delta_{\mu_2 4}\delta_{\nu_2 4},\\B^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}&=B^{\mu_1\nu_1\mu_2\nu_2}P^{\mu_3\nu_3}\\&\quad+\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2}\delta_{\mu_3 4}\delta_{\nu_3 4},\cdots,\\B^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n\mu_n\nu_n}&=B^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_{n-1}\nu_{n-1}}P^{\mu_n\nu_n}\\&\quad+\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2}\cdots\delta_{\mu_{n-1}\nu_{n-1}}\delta_{\mu_n 4}\delta_{\nu_n 4}.\end{aligned}\tag{75}$$

Equation (73b) gives the general momentum representation for the Feynman propagator for an arbitrary integral spin. As an application of these formulas, we give finally an explicit expression for the Feynman propagator for spin 4 in momentum representation:

$$\begin{aligned}&\Delta_{\mathrm{F}}^{\mu_1\mu_2\mu_3\mu_4\nu_1\nu_2\nu_3\nu_4}(4,p)\\&=P^{\mu_1\mu_2\mu_3\mu_4\nu_1\nu_2\nu_3\nu_4}(4,p)\frac{-\mathrm{i}}{p^2+W^2-\mathrm{i}\varepsilon}\\&\quad+K^{\mu_1\mu_2\mu_3\mu_4\nu_1\nu_2\nu_3\nu_4}(4,p),\end{aligned}\tag{76}$$

where

$$\begin{aligned}&P^{\mu_1\mu_2\mu_3\mu_4\nu_1\nu_2\nu_3\nu_4}(4,p)\\&=\left(\frac{1}{4!}\right)^2\sum_{\substack{P(\mu_1\mu_2\mu_3\mu_4)\\P(\nu_1\nu_2\nu_3\nu_4)}}\Big[P^{\mu_1\nu_1}P^{\mu_2\nu_2}P^{\mu_3\nu_3}\,P^{\mu_4\nu_4}\\&\quad-\frac{6}{7}P^{\mu_1\mu_2}P^{\nu_1\nu_2}P^{\mu_3\nu_3}P^{\mu_4\nu_4}\\&\quad+\frac{3}{35}P^{\mu_1\mu_2}P^{\nu_1\nu_2}P^{\mu_3\mu_4}P^{\nu_3\nu_4}\Big],\end{aligned}$$

and

$$\begin{aligned}&K^{\mu_1\mu_2\mu_3\mu_4\nu_1\nu_2\nu_3\nu_4}(4,p)\\&=\frac{\mathrm{i}}{(4!W)^2}\sum_{\substack{P(\mu_1\mu_2\mu_3\mu_4)\\P(\nu_1\nu_2\nu_3\nu_4)}}\Big[B^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\\&\quad-\frac{6}{7}B^{\mu_1\mu_2\nu_1\nu_2\mu_3\nu_3\mu_4\nu_4}+\frac{3}{35}B^{\mu_1\mu_2\nu_1\nu_2\mu_3\mu_4\nu_3\nu_4}\Big].\end{aligned}$$

3.2 Half-integral spin

We now proceed to derive the Feynman propagator for a particle with arbitrary half-integral spin. We begin with working out the Feynman propagators for spin 3/2 and 5/2 in detail in order to demonstrate the method employed in this section. After that, the procedure is generalized to the case for an arbitrary half-integral spin $n+1/2$.

3.2.1 Spin 3/2

In the case of spin 3/2, armed with the wave functions

$$\begin{aligned}\Psi^\nu(x) &= \frac{1}{\sqrt{V}}\sum_{\boldsymbol{p}}\sum_{m=-3/2}^{3/2}\left[a_m(\boldsymbol{p})U_m^\nu(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}\right. \\ &\quad \left.+b_m^+(\boldsymbol{p})V_m^\nu(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}\right] \\ &= \Psi^\nu(x)^{(-)}+\Psi^\nu(x)^{(+)}, \end{aligned} \tag{77}$$

the projection operator $P^{\mu\nu}\left(\frac{3}{2},p\right)$ given by (29a), the field quantization conditions

$$\left\{a_m(\boldsymbol{p}),a_{m'}^+(\boldsymbol{p}')\right\} = \delta_{\boldsymbol{p},\boldsymbol{p}'}\delta_{mm'} = \left\{b_m(\boldsymbol{p}),b_{m'}^+(\boldsymbol{p}')\right\}, \tag{78}$$

with all others vanishing, together with the definition of the adjoint field

$$\begin{aligned}\bar{\Psi}^\mu(x) &= \left(\Psi^\mu(x)\right)^+\gamma_4 \\ &= \frac{1}{\sqrt{V}}\sum_{\boldsymbol{p}}\sum_{m=-3/2}^{3/2}\left[a_m^+(\boldsymbol{p})\bar{U}_m^\mu(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}+b_m^(\boldsymbol{p})\bar{V}_m^\mu(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}\right] \\ &= \bar{\Psi}^\mu(x)^{(+)}+\bar{\Psi}^\mu(x)^{(-)}, \end{aligned} \tag{79}$$

we find that the general commutation relations take the following form:

$$\left\{\Psi^\mu(x),\bar{\Psi}^\nu(x')\right\} = \mathrm{i}\hat{P}^{\mu\nu}\left(\frac{3}{2}\right)\Delta(x-x'), \tag{80}$$

or equivalently

$$\left\{\Psi^\mu(x)^{(-)},\bar{\Psi}^\nu(x')^{(+)}\right\} = \mathrm{i}\hat{P}^{\mu\nu}\left(\frac{3}{2}\right)\Delta^{(+)}(x-x'), \tag{81a}$$

$$\left\{\Psi^\mu(x)^{(+)},\bar{\Psi}^\nu(x')^{(-)}\right\} = \mathrm{i}\hat{P}^{\mu\nu}\left(\frac{3}{2}\right)\Delta^{(-)}(x-x'), \tag{81b}$$

where

$$\hat{P}^{\mu\nu}\left(\frac{3}{2}\right) = -\left(\partial\!\!\!/-W\right)\hat{R}^{\mu\nu}\left(\frac{3}{2}\right), \tag{82a}$$

$$\hat{R}^{\mu\nu}\left(\frac{3}{2}\right) = \hat{P}_{\mu\nu} - \frac{1}{3}\left[\gamma_\mu\gamma_\nu-\frac{1}{W}\left(\gamma_\mu\partial_\nu-\gamma_\nu\partial_\mu\right)-\frac{1}{W^2}\partial_\mu\partial_\nu\right]. \tag{82b}$$

The Feynman propagator for spin 3/2 is in the usual fashion defined by

$$\begin{aligned}S_{\mathrm{F}}^{\mu\nu}(x-x') &\equiv \langle 0|\,T\Psi^\mu(x)\bar{\Psi}^\nu(x')|0\rangle \\ &= \begin{cases}\langle 0|\,\Psi^\mu(x)\bar{\Psi}^\nu(x')|0\rangle & t>t', \\ -\langle 0|\,\bar{\Psi}^\nu(x')\Psi^\mu(x)|0\rangle & t<t',\end{cases}\end{aligned} \tag{83}$$

and can be rewritten as, with the aid of (81),

$$\begin{aligned}S_{\mathrm{F}}^{\mu\nu}(x-x') &= \mathrm{i}\theta(t-t')\hat{P}^{\mu\nu}\left(\frac{3}{2}\right)\Delta^{(+)}(x-x') \\ &\quad -\mathrm{i}\theta(t'-t)\hat{P}^{\mu\nu}\left(\frac{3}{2}\right)\Delta^{(-)}(x-x').\end{aligned} \tag{84}$$

Performing a direct calculation that makes the θ functions commuted past the differential operator $\hat{P}^{\mu\nu}\left(\frac{3}{2}\right)$, with the help of (41), one finds that the Feynman propagator for spin 3/2 in coordinate representation takes the form

$$\begin{aligned}S_{\mathrm{F}}^{\mu\nu}(x-x') &= \hat{P}^{\mu\nu}\left(\frac{3}{2}\right)\Delta_{\mathrm{F}}(x-x') \\ &\quad +\hat{K}^{\mu\nu}\left(\frac{3}{2}\right)\delta^{(4)}(x-x'),\end{aligned} \tag{85}$$

where

$$\begin{aligned}\hat{K}^{\mu\nu}\left(\frac{3}{2}\right) &= -\mathrm{i}\left[\frac{1}{3W}\gamma_4\left(\delta_{\mu4}\gamma_\nu-\gamma_\mu\delta_{\nu4}\right)\right. \\ &\quad \left.+\frac{2}{3W^2}\left(\partial\!\!\!/-W\right)\delta_{\mu4}\delta_{\nu4}\right].\end{aligned} \tag{86}$$

Utilizing (45), the Fourier representation for $S_{\mathrm{F}}^{\mu\nu}(x)$ is found to be

$$S_{\mathrm{F}}^{\mu\nu}(x) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4p\,\mathrm{e}^{\mathrm{i}px}S_{\mathrm{F}}^{\mu\nu}\left(\frac{3}{2},p\right), \tag{87}$$

$$\begin{aligned}S_{\mathrm{F}}^{\mu\nu}\left(\frac{3}{2},p\right) &= \frac{-1}{p\!\!\!/-\mathrm{i}W+\mathrm{i}\varepsilon}R^{\mu\nu}\left(\frac{3}{2},p\right) \\ &\quad +K^{\mu\nu}\left(\frac{3}{2},p\right),\end{aligned} \tag{88}$$

with

$$\begin{aligned}&R^{\mu\nu}\left(\frac{3}{2},p\right) \\ &= P_{\mu\nu}-\frac{1}{3}\left[\gamma_\mu\gamma_\nu-\frac{\mathrm{i}}{W}\left(\gamma_\mu p_\nu-\gamma_\nu p_\mu\right)-\frac{1}{W^2}p_\mu p_\nu\right],\end{aligned} \tag{89a}$$

$$\begin{aligned}&K^{\mu\nu}\left(\frac{3}{2},p\right) \\ &= -\mathrm{i}\left[\frac{1}{3W}\gamma_4\left(\delta_{\mu4}\gamma_\nu-\gamma_\mu\delta_{\nu4}\right)+\frac{2}{3W^2}\left(\mathrm{i}\,p\!\!\!/-W\right)\delta_{\mu4}\delta_{\nu4}\right];\end{aligned} \tag{89b}$$

$S_{\mathrm{F}}^{\mu\nu}\left(\frac{3}{2},p\right)$ is the Feynman propagator for spin 3/2 in momentum representation.

3.2.2 Spin 5/2

In a similar manner as in the case of spin 3/2, by using the wave functions for spin 5/2

$$\begin{aligned}\Psi^{\nu_1\nu_2}(x) &= \sum_{\boldsymbol{p}} \sum_{m=-5/2}^{5/2} \frac{1}{\sqrt{V}} \left[a_m(\boldsymbol{p})U_m^{\nu_1\nu_2}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}\right.\\ &\quad \left.+b_m^+(\boldsymbol{p})V_m^{\nu_1\nu_2}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}\right]\\ &= \Psi^{\nu_1\nu_2}(x)^{(-)} + \Psi^{\nu_1\nu_2}(x)^{(+)}, \end{aligned} \tag{90}$$

the projection operator $P^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2},p\right)$ given by (29b), the field quantization conditions (78), and the definition of the adjoint field

$$\begin{aligned}\bar{\Psi}^{\nu_1\nu_2}(x) &= \left(\Psi^{\nu_1\nu_2}(x)\right)^+\gamma_4\\ &= \sum_{\boldsymbol{p}} \sum_{m=-5/2}^{5/2} \frac{1}{\sqrt{V}} \left[a_m^+(\boldsymbol{p})\bar{U}_m^{\nu_1\nu_2}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px}\right.\\ &\quad \left.+b_m(\boldsymbol{p})\bar{V}_m^{\nu_1\nu_2}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}\right]\\ &= \bar{\Psi}^{\nu_1\nu_2}(x)^{(+)} + \bar{\Psi}^{\nu_1\nu_2}(x)^{(-)}, \end{aligned} \tag{91}$$

we find that the general commutation rules are

$$\left\{\Psi^{\mu_1\mu_2}(x), \bar{\Psi}^{\nu_1\nu_2}(x')\right\} = \mathrm{i}\hat{P}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right)\Delta(x-x'), \tag{92}$$

or equivalently

$$\begin{aligned}&\left\{\Psi^{\mu_1\mu_2}(x)^{(-)}, \bar{\Psi}^{\nu_1\nu_2}(x')^{(+)}\right\}\\ &= \mathrm{i}\hat{P}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right)\Delta^{(+)}(x-x'),\end{aligned} \tag{93a}$$

$$\begin{aligned}&\left\{\Psi^{\mu_1\mu_2}(x)^{(+)}, \bar{\Psi}^{\nu_1\nu_2}(x')^{(-)}\right\}\\ &= \mathrm{i}\hat{P}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right)\Delta^{(-)}(x-x'),\end{aligned} \tag{93b}$$

where

$$\hat{P}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right) = -(\partial\!\!\!/ - W)\hat{R}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right), \tag{94a}$$

$$\begin{aligned}&\hat{R}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right)\\ &= \frac{1}{2}\left[\hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2} + \hat{P}^{\mu_1\nu_2}\hat{P}^{\mu_2\nu_1}\right] - \frac{1}{5}\left[\hat{P}^{\mu_1\mu_2}\hat{P}^{\nu_1\nu_2}\right]\\ &\quad - \frac{1}{10}\sum_{\substack{P(\mu_1\mu_2)\\P(\nu_1\nu_2)}}\left[\gamma_{\mu_1}\gamma_{\nu_1}\right.\\ &\quad \left.-\frac{1}{W}\left(\gamma_{\mu_1}\partial_{\nu_1} - \gamma_{\nu_1}\partial_{\mu_1}\right) - \frac{1}{W^2}\partial_{\mu_1}\partial_{\nu_1}\right]\hat{P}^{\mu_2\nu_2}.\end{aligned} \tag{94b}$$

The Feynman propagator for spin 5/2 is defined by

$$S_{\mathrm{F}}^{\mu_1\mu_2\nu_1\nu_2}(x-x') \equiv \langle 0|\, T\Psi^{\mu_1\mu_2}(x)\bar{\Psi}^{\nu_1\nu_2}(x')|0\rangle \tag{95}$$

$$= \begin{cases}\langle 0|\Psi^{\mu_1\mu_2}(x)\bar{\Psi}^{\nu_1\nu_2}(x')|0 \\ \qquad t > t', \\ -\langle 0|\bar{\Psi}^{\nu_1\nu_2}(x')\Psi^{\mu_1\mu_2}(x)|0\rangle \\ \qquad t < t',\end{cases}$$

and can be re-expressed, with the aid of (93), as

$$\begin{aligned}&S_{\mathrm{F}}^{\mu_1\mu_2\nu_1\nu_2}(x-x')\\ &= \mathrm{i}\theta(t-t')\hat{P}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right)\Delta^{(+)}(x-x')\\ &\quad -\mathrm{i}\theta(t'-t)\hat{P}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right)\Delta^{(-)}(x-x').\end{aligned} \tag{96}$$

Performing a direct calculation that makes the θ functions commuted past the differential operator $\hat{P}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right)$, we get

$$\begin{aligned}&S_{\mathrm{F}}^{\mu_1\mu_2\nu_1\nu_2}(x-x')\\ &= \hat{P}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right)\Delta_{\mathrm{F}}(x-x')\\ &\quad +\hat{K}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right)\delta^{(4)}(x-x'),\end{aligned} \tag{97}$$

where

$$\begin{aligned}&\hat{K}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2}\right)\\ &= -\frac{\mathrm{i}(\partial\!\!\!/ - W)}{W^2}\left[\frac{1}{2}\left(\delta_{\mu_1\nu_1}\delta_{\mu_2 4}\delta_{\nu_2 4}\right.\right.\\ &\quad \left.+\delta_{\mu_1 4}\delta_{\nu_1 4}\hat{P}^{\mu_2\nu_2} + \delta_{\mu_1\nu_2}\delta_{\mu_2 4}\delta_{\nu_1 4} + \delta_{\mu_1 4}\delta_{\nu_2 4}\hat{P}^{\mu_2\nu_1}\right)\\ &\quad \left.-\frac{1}{5}\left(\delta_{\mu_1\mu_2}\delta_{\nu_1 4}\delta_{\nu_2 4} + \delta_{\mu_1 4}\delta_{\mu_2 4}\hat{P}^{\nu_1\nu_2}\right)\right]\\ &\quad -\frac{1}{10}\sum_{\substack{P(\mu_1\mu_2)\\P(\nu_1\nu_2)}}\left\{\left[\frac{\mathrm{i}}{W}\gamma_4\left(\delta_{\mu_1 4}\gamma_{\nu_1} - \gamma_{\mu_1}\delta_{\nu_1 4}\right)\right.\right.\\ &\quad \left.-\frac{\mathrm{i}}{W^2}\left(\partial\!\!\!/ - W\right)\delta_{\mu_1 4}\delta_{\nu_1 4}\right]\hat{P}^{\mu_2\nu_2}\\ &\quad \left.+(\partial\!\!\!/ - W)\gamma_{\mu_1}\gamma_{\nu_1}\left(\frac{\delta_{\mu_2 4}\delta_{\nu_2 4}}{W^2}\right)\right\}.\end{aligned} \tag{98}$$

This is the Feynman propagator for spin 5/2 in coordinate representation. The Fourier representation for $S_{\mathrm{F}}^{\mu_1\mu_2\nu_1\nu_2}(x)$ can be easily deduced, using (45),

$$S_{\mathrm{F}}^{\mu_1\mu_2\nu_1\nu_2}(x) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4p\mathrm{e}^{\mathrm{i}px}S_{\mathrm{F}}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2},p\right), \tag{99}$$

$$\begin{aligned}&S_{\mathrm{F}}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2},p\right)\\ &= \frac{-1}{p\!\!\!/ - \mathrm{i}W + \mathrm{i}\varepsilon}R^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2},p\right) + K^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2},p\right),\end{aligned} \tag{100}$$

with

$$R^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2},p\right)$$

$$= \frac{1}{2}\left[P^{\mu_1\nu_1}P^{\mu_2\nu_2} + P^{\mu_1\nu_2}P^{\mu_2\nu_1}\right] - \frac{1}{5}\left[P^{\mu_1\mu_2}P^{\nu_1\nu_2}\right] - \frac{1}{10}\sum_{\substack{P(\mu_1\mu_2)\\P(\nu_1\nu_2)}}\left[\gamma_{\mu_1}\gamma_{\nu_1} - \frac{\mathrm{i}}{W}\left(\gamma_{\mu_1}p_{\nu_1} - \gamma_{\nu_1}p_{\mu_1}\right) - \frac{1}{W^2}p_{\mu_1}p_{\nu_1}\right]P^{\mu_2\nu_2}, \tag{101a}$$

$$K^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2},p\right) = -\frac{\mathrm{i}(\mathrm{i}\not{p}-W)}{W^2}\left[\frac{1}{2}\left(\delta_{\mu_1\nu_1}\delta_{\mu_2 4}\delta_{\nu_2 4} + \delta_{\mu_1 4}\delta_{\nu_1 4}P^{\mu_2\nu_2} + \delta_{\mu_1\nu_2}\delta_{\mu_2 4}\delta_{\nu_1 4} + \delta_{\mu_1 4}\delta_{\nu_2 4}P^{\mu_2\nu_1}\right) - \frac{1}{5}\left(\delta_{\mu_1\mu_2}\delta_{\nu_1 4}\delta_{\nu_2 4} + \delta_{\mu_1 4}\delta_{\mu_2 4}P^{\nu_1\nu_2}\right)\right] - \frac{1}{10}\sum_{\substack{P(\mu_1\mu_2)\\P(\nu_1\nu_2)}}\left\{\left[\frac{\mathrm{i}}{W}\gamma_4\left(\delta_{\mu_1 4}\gamma_{\nu_1} - \gamma_{\mu_1}\delta_{\nu_1 4}\right) - \frac{\mathrm{i}}{W^2}\left(\mathrm{i}\not{p}-W\right)\delta_{\mu_1 4}\delta_{\nu_1 4}\right]P^{\mu_2\nu_2} + (\mathrm{i}\not{p}-W)\gamma_{\mu_1}\gamma_{\nu_1}\left(\frac{\delta_{\mu_2 4}\delta_{\nu_2 4}}{W^2}\right)\right\}. \tag{101b}$$

Similar to the previous case, $S_{\mathrm{F}}^{\mu_1\mu_2\nu_1\nu_2}\left(\frac{5}{2},p\right)$ in the above is the Feynman propagator for spin 5/2 in momentum representation.

3.2.3 Spin $n+1/2$

The above procedure is now extended to the general case of arbitrary half-integral spin $n+1/2$. We begin by working out the general commutation rules. From (10), (12), (15), (23) and (78), we get, by following the same steps as in the cases of spin 3/2 and 5/2:

$$\left\{\Psi^{\mu_1\mu_2\cdots\mu_n}(x), \bar{\Psi}^{\nu_1\nu_2\cdots\nu_n}(x')\right\} = \mathrm{i}\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right)\Delta(x-x'), \tag{102}$$

or equivalently

$$\left\{\Psi^{\mu_1\mu_2\cdots\mu_n}(x)^{(-)}, \bar{\Psi}^{\nu_1\nu_2\cdots\nu_n}(x')^{(+)}\right\} = \mathrm{i}\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right)\Delta^{(+)}(x-x'), \tag{103a}$$

$$\left\{\Psi^{\mu_1\mu_2\cdots\mu_n}(x)^{(+)}, \bar{\Psi}^{\nu_1\nu_2\cdots\nu_n}(x')^{(-)}\right\} = \mathrm{i}\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right)\Delta^{(-)}(x-x'), \tag{103b}$$

with

$$\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right) = -(\not{\partial}-W)\hat{R}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right), \tag{104}$$

where $\hat{R}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}\left(n+\frac{1}{2}\right)$ is a differential operator constructed from $Q^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}\left(n+\frac{1}{2},p\right)$ in the following way (refer to (28)):

$$3P^{\mu_1\nu_1}P^{\mu_2\nu_2}\cdots P^{\mu_n\nu_n} \Rightarrow 3\hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}\cdots\hat{P}^{\mu_n\nu_n}, \tag{105}$$

$$\left[\gamma_{\mu_1}\gamma_{\nu_1} \mp \frac{\mathrm{i}}{W}\left(\gamma_{\mu_1}p_{\nu_1} - \gamma_{\nu_1}p_{\mu_1}\right) + \frac{1}{W^2}p_{\mu_1}p_{\nu_1}\right]\times P^{\mu_2\nu_2}\cdots P^{\mu_n\nu_n} \Rightarrow \left[\gamma_{\mu_1}\gamma_{\nu_1} - \frac{1}{W}\left(\gamma_{\mu_1}\partial_{\nu_1} - \gamma_{\nu_1}\partial_{\mu_1}\right) - \frac{1}{W^2}\partial_{\mu_1}\partial_{\nu_1}\right]\times\hat{P}^{\mu_2\nu_2}\cdots\hat{P}^{\mu_n\nu_n}. \tag{106}$$

The Feynman propagator for spin $n+1/2$ is defined by

$$S_{\mathrm{F}}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(x-x') \equiv \langle 0|T\Psi^{\mu_1\mu_2\cdots\mu_n}(x)\bar{\Psi}^{\nu_1\nu_2\cdots\nu_n}(x')|0\rangle = \begin{cases}\langle 0|\,\Psi^{\mu_1\mu_2\cdots\mu_n}(x)\bar{\Psi}^{\nu_1\nu_2\cdots\nu_n}(x')|0\rangle & t>t',\\ -\langle 0|\,\bar{\Psi}^{\nu_1\nu_2\cdots\nu_n}(x')\Psi^{\mu_1\mu_2\cdots\mu_n}(x)|0\rangle & t<t',\end{cases} \tag{107}$$

and can be rewritten as, using (103),

$$S_{\mathrm{F}}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(x-x') = \mathrm{i}\theta(t-t')\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right)\Delta^{(+)}(x-x') - \mathrm{i}\theta(t'-t)\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right)\Delta^{(-)}(x-x'). \tag{108}$$

Performing a step-by-step calculation that makes the θ functions commute past the differential operator $\hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right)$, we find that the general expression for the Feynman propagator for spin $n+1/2$ in coordinate representation is

$$S_{\mathrm{F}}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(x-x') = \hat{P}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right)\Delta_{\mathrm{F}}(x-x') + \hat{K}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right)\delta^{(4)}(x-x'), \tag{109}$$

where $\hat{K}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}\left(n+\frac{1}{2}\right)$ is resulting from the commutation between the θ functions and $\hat{P}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}\left(n+\frac{1}{2}\right)$, and consists of two kinds of terms [refer to (105) and (106)]; one originates from terms such as $(\not{\partial}-W)3\hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}\cdots\hat{P}^{\mu_n\nu_n}$, and can be expressed as $\frac{3\mathrm{i}(\not{\partial}-W)}{W^2}\left[\hat{B}^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}\right]$, because of

$$\mathrm{i}\theta(t)\left(\not{\partial}-W\right)\hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}\cdots\hat{P}^{\mu_n\nu_n}\Delta^{(+)}(x) - \mathrm{i}\theta(-t)\left(\not{\partial}-W\right)\hat{P}^{\mu_1\nu_1} - \mathrm{i}\theta(-t)\left(\not{\partial}-W\right)\hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}\cdots\hat{P}^{\mu_n\nu_n}\Delta^{(-)}(x) = \left(\not{\partial}-W\right)\hat{P}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}\cdots\hat{P}^{\mu_n\nu_n}\Delta_{\mathrm{F}}(x) + \frac{\mathrm{i}\left(\not{\partial}-W\right)}{W^2}\left[\hat{B}^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}\right]\delta^{(4)}(x), \tag{110}$$

where $\hat{B}^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}$ is the same operator as in the case of integral spin (see (71)), the other originates from terms such as

$$(\not{\partial}-W)\times\left[\gamma_{\mu_1}\gamma_{\nu_1}-\frac{1}{W}\left(\gamma_{\mu_1}\partial_{\nu_1}-\gamma_{\nu_1}\partial_{\mu_1}\right)-\frac{1}{W^2}\partial_{\mu_1}\partial_{\nu_1}\right]\times\hat{P}^{\mu_2\nu_2}\ldots\hat{P}^{\mu_n\nu_n},$$

and can be expressed by $\hat{C}^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}$, because of

$$\begin{aligned}
&\mathrm{i}\theta(t)\,(\not{\partial}-W)\left[\gamma_{\mu_1}\gamma_{\nu_1}-\frac{1}{W}\left(\gamma_{\mu_1}\partial_{\nu_1}-\gamma_{\nu_1}\partial_{\mu_1}\right)-\frac{1}{W^2}\partial_{\mu_1}\partial_{\nu_1}\right]\\
&\quad\times\hat{P}^{\mu_2\nu_2}\cdots\hat{P}^{\mu_n\nu_n}\Delta^{(+)}(x)-\mathrm{i}\theta(-t)\,(\not{\partial}-W)\\
&\quad\times\left[\gamma_{\mu_1}\gamma_{\nu_1}-\frac{1}{W}\left(\gamma_{\mu_1}\partial_{\nu_1}-\gamma_{\nu_1}\partial_{\mu_1}\right)-\frac{1}{W^2}\partial_{\mu_1}\partial_{\nu_1}\right]\\
&\quad\times\hat{P}^{\mu_2\nu_2}\ldots\hat{P}^{\mu_n\nu_n}\Delta^{(-)}(x)\\
&=(\not{\partial}-W)\left[\gamma_{\mu_1}\gamma_{\nu_1}-\frac{1}{W}\left(\gamma_{\mu_1}\partial_{\nu_1}-\gamma_{\nu_1}\partial_{\mu_1}\right)-\frac{1}{W^2}\partial_{\mu_1}\partial_{\nu_1}\right]\\
&\quad\times\hat{P}^{\mu_2\nu_2}\ldots\hat{P}^{\mu_n\nu_n}\Delta_{\mathrm{F}}(x)\\
&\qquad+\hat{C}^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}\delta^{(4)}(x),
\end{aligned}\tag{111a}$$

with

$$\begin{aligned}
&\hat{C}^{\mu_1\nu_1}\\
&=\mathrm{i}\left[\frac{1}{W}\gamma_4\left(\delta_{\mu_1 4}\gamma_{\nu_1}-\gamma_{\mu_1}\delta_{\nu_1 4}\right)-\frac{1}{W^2}\left(\not{\partial}-W\right)\delta_{\mu_1 4}\delta_{\nu_1 4}\right],\\
&\hat{C}^{\mu_1\nu_1\mu_2\nu_2}\\
&=\hat{C}^{\mu_1\nu_1}\hat{P}^{\mu_2\nu_2}+\mathrm{i}(\not{\partial}-W)\gamma_{\mu_1}\gamma_{\nu_1}\left(\frac{\delta_{\mu_2 4}\delta_{\nu_2 4}}{W^2}\right),\\
&\hat{C}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\\
&=\hat{C}^{\mu_1\nu_1\mu_2\nu_2}\hat{P}^{\mu_3\nu_3}\\
&\qquad+\mathrm{i}(\not{\partial}-W)\gamma_{\mu_1}\gamma_{\nu_1}\delta_{\mu_2\nu_2}\left(\frac{\delta_{\mu_3 4}\delta_{\nu_3 4}}{W^2}\right),\cdots,\\
&\hat{C}^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}\\
&=\hat{C}^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_{n-1}\nu_{n-1}}\hat{P}^{\mu_n\nu_n}\\
&\qquad+\mathrm{i}(\not{\partial}-W)\gamma_{\mu_1}\gamma_{\nu_1}\delta_{\mu_2\nu_2}\delta_{\mu_3\nu_3}\cdots\delta_{\mu_{n-1}\nu_{n-1}}\left(\frac{\delta_{\mu_n 4}\delta_{\nu_n 4}}{W^2}\right).
\end{aligned}\tag{111b}$$

The Fourier representation for $S_{\mathrm{F}}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(x)$ can be easily derived, using (45),

$$S_{\mathrm{F}}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(x)=\frac{1}{(2\pi)^4}\int\mathrm{d}^4p\,\mathrm{e}^{\mathrm{i}px}S_{\mathrm{F}}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2},p\right),\tag{112a}$$

$$\begin{aligned}
&S_{\mathrm{F}}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2},p\right)\\
&=\frac{-1}{\not{p}-\mathrm{i}W+\mathrm{i}\varepsilon}R^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2},p\right)\\
&\qquad+K^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2},p\right),
\end{aligned}\tag{112b}$$

with

$$R^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2},p\right)=\hat{R}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right)\Big|_{\partial_\mu=\mathrm{i}p_\mu},\tag{113a}$$

$$K^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2},p\right)=\hat{K}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\left(n+\frac{1}{2}\right)\Big|_{\partial_\mu=\mathrm{i}p_\mu}.\tag{113b}$$

Equation (112b) gives the general momentum representation for the Feynman propagator for an arbitrary half-integral spin. As an illustration of these formulas, we provide finally the explicit expression for the propagator for spin 7/2:

$$\begin{aligned}
&S_{\mathrm{F}}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\left(\frac{7}{2},p\right)\\
&=\frac{-1}{\not{p}-\mathrm{i}W+\mathrm{i}\varepsilon}R^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\left(\frac{7}{2},p\right)\\
&\qquad+K^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\left(\frac{7}{2},p\right),
\end{aligned}\tag{114}$$

where

$$\begin{aligned}
&R^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\left(\frac{7}{2},p\right)\\
&=\frac{1}{6}\sum_{P(\nu_1\nu_2\nu_3)}P^{\mu_1\nu_1}P^{\mu_2\nu_2}P^{\mu_3\nu_3}\\
&\quad-\frac{1}{280}\sum_{\substack{P(\mu_1\mu_2\mu_3)\\P(\nu_1\nu_2\nu_3)}}P^{\mu_1\mu_2}P^{\nu_1\nu_2}P^{\mu_3\nu_3}\\
&\quad-\frac{1}{84}\sum_{\substack{P(\mu_1\mu_2\mu_3)\\P(\nu_1\nu_2\nu_3)}}\Bigg[\Bigg(\gamma_{\mu_1}\gamma_{\nu_1}-\frac{\mathrm{i}}{W}\left(\gamma_{\mu_1}p_{\nu_1}-\gamma_{\nu_1}p_{\mu_1}\right)\\
&\qquad+\frac{1}{W^2}p_{\mu_1}p_{\nu_1}\Bigg)P^{\mu_2\nu_2}P^{\mu_3\nu_3}\\
&\quad+\frac{2}{9}\Bigg(\gamma_{\mu_1}\gamma_{\mu_2}-\frac{\mathrm{i}}{W}\left(\gamma_{\mu_1}p_{\mu_2}-\gamma_{\mu_2}p_{\mu_1}\right)\\
&\qquad+\frac{1}{W^2}p_{\mu_1}p_{\mu_2}\Bigg)P^{\nu_1\nu_2}P^{\mu_3\nu_3}\\
&\quad+\frac{1}{5}\Bigg(\gamma_{\nu_1}\gamma_{\mu_1}-\frac{\mathrm{i}}{W}\left(\gamma_{\nu_1}p_{\mu_1}-\gamma_{\mu_1}p_{\nu_1}\right)\\
&\qquad+\frac{1}{W^2}p_{\mu_1}p_{\nu_1}\Bigg)P^{\mu_2\mu_3}P^{\nu_2\nu_3}\Bigg]\Bigg\},
\end{aligned}\tag{115a}$$

$$\begin{aligned}
&K^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\left(\frac{7}{2},p\right)\\
&=\frac{1}{W^2}\left(\not{p}+\mathrm{i}W\right)\\
&\quad\times\Bigg[\frac{1}{6}\sum_{P(\nu_1\nu_2\nu_3)}B^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}
\end{aligned}$$

$$- \frac{1}{280} \sum_{\substack{P(\mu_1\mu_2\mu_3) \\ P(\nu_1\nu_2\nu_3)}} B^{\mu_1\mu_2\nu_1\nu_2\mu_3\nu_3} \Bigg]$$

$$+\frac{1}{84} \sum_{\substack{P(\mu_1\mu_2\mu_3) \\ P(\nu_1\nu_2\nu_3)}} \Big[C^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}$$

$$+\frac{2}{9}C^{\mu_1\mu_2\nu_1\nu_2\mu_3\nu_3} + \frac{1}{5}C^{\nu_1\mu_1\mu_2\mu_3\nu_2\nu_3}\Big], \qquad (115b)$$

with

$$C^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} \qquad (116a)$$
$$= C^{\mu_1\nu_1\mu_2\nu_2} P^{\mu_3\nu_3} + \frac{\mathrm{i}(\mathrm{i}\, \not{p} - W)}{W^2} \left[\gamma_{\mu_1}\gamma_{\nu_1}\delta_{\mu_2\nu_2}\delta_{\mu_3 4}\delta_{\nu_3 4}\right],$$
$$C^{\mu_1\nu_1\mu_2\nu_2} \qquad (116b)$$
$$= C^{\mu_1\nu_1} P^{\mu_2\nu_2} + \frac{\mathrm{i}(\mathrm{i}\, \not{p} - W)}{W^2} \left[\gamma_{\mu_1}\gamma_{\nu_1}\delta_{\mu_2 4}\delta_{\nu_2 4}\right],$$
$$C^{\mu_1\nu_1} \qquad (116c)$$
$$= \mathrm{i}\left[\frac{1}{W}\gamma_4\left(\delta_{\mu_1 4}\gamma_{\nu_1} - \gamma_{\mu_1}\delta_{\nu_1 4}\right) - \frac{(\mathrm{i}\, \not{p} - W)}{W^2}\delta_{\mu_1 4}\delta_{\nu_1 4}\right].$$

In summary, the projection operator for an arbitrary integral and half-integral spin constructed by Behrends and Fronsdal has been confirmed and simplified by direct derivation based on the explicit expression of the wave functions, the commutation rules and a general expression for the Feynman propagator for an arbitrary integral and half-integral spin are deduced, and especially explicit expressions for the propagators for spin 3/2, 2, 5/2, 3, 7/2 and 4 are provided.

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